* Recall

$(H, \mu, \iota, \Delta, \varepsilon, S, S^{-1})$: Hopf algebra, finite dimensional, invertible antipode

$\Rightarrow X = (H^{op})^* = (H^*, \Delta^*, \varepsilon^*, (\mu^{op})^*, \iota^*, (S^{-1})^*, S^*)$: Hopf algebra

And (X, H) is matched. $\Rightarrow$ $D(H) = X \bowtie H$ is the quantum double.

$D(H)$ ~~has~~ is a vector space $X \otimes H$.

(From now on, we write $f \otimes a$ by fa, if no confusion arise)

$D(H)$ has a Hopf algebra structure as follows:

[multiplication] $(fa)(gb) = \sum_{(a)} \underbrace{f g(S^{-1}(a''')\_\, a')}_{X} \underbrace{a'' b}_{H}$

[Unit] $1 = 1_X 1_H$. Note that $1_X = \varepsilon^* : k^* \simeq k \longrightarrow H^*$, $1 \longmapsto \varepsilon$

[Comultiplication] $\Delta(fa) = \sum_{(a),(f)} f'a' \otimes f''a''$

[Counit] $\varepsilon(fa) = f(1)\,\varepsilon(a)$

[Antipode] $S(fa) = \sum_{(a)} f(a' S^{-1}(\_)\, S^{-1}(a''')) \, S(a'')$

Recall On X; $\langle u, fg \rangle := \sum_{(u)} \langle u', f \rangle \langle u'', g \rangle$ for $f, g \in X$

$\varepsilon(f) = f(1)$, $\iota(1) = \varepsilon \in X$

$\Delta(f) = \sum f' \otimes f'' \iff \langle uv, f \rangle = \sum_{(f)} \langle v, f' \rangle \langle u, f'' \rangle$.

$\boxed{\text{Thm}}$ $R=\sum_{i=1}^{n} e_i\otimes e^i$ is the universal R-matrix if

$n=\dim H$ and $\{e_i\}$ is a basis of H, $\{e^i\}$ is a dual basis in X.

⟨pf⟩ Put $\bar{R}=\sum_{j=1}^{n} e_j\otimes e^j\circ S$

$$\langle af\otimes bg,\ \varphi x\otimes \psi y\rangle$$
$$=\langle af,\varphi x\rangle\langle bg,\psi y\rangle$$
$$=\langle a,\varphi\rangle\langle x,f\rangle\langle b,\psi\rangle\langle y,g\rangle$$

① $R\bar{R}=1$

$R\bar{R}=\sum_{i,j} e_ie_j\otimes e^i(e^j\circ S)$. Now evaluate at $af\otimes bg$

$$\langle af\otimes bg,\ R\bar{R}\rangle=\sum_{i,j}\langle e_ie_j,f\rangle\langle b,e^i(e^j\circ S)\rangle\,\varepsilon(a)g(1)$$

$$=\sum_{\substack{i,j\\(f),(b)}}\langle e_j,f'\rangle\langle e_i,f''\rangle\langle b',e^i\rangle\langle b'',e^j\circ S\rangle\,\varepsilon(a)g(1)$$

$$=\sum_{\substack{i,j\\(f),(b)}}\langle b',\langle e_i,f''\rangle e^i\rangle\langle e_j,f'\rangle\langle S(b''),e^j\rangle\,\varepsilon(a)g(1)$$

$$=\sum_{\substack{j\\(f),(b)}}\langle b',f''\rangle\langle S(b''),\langle e_j,f'\rangle e^j\rangle\,\varepsilon(a)g(1)$$

$$=\sum_{(f),(c)}\langle b',f''\rangle\langle S(b''),f'\rangle=\sum_{(b)}\langle b'S(b''),f\rangle\,\varepsilon(a)g(1)$$

$$\cancel{\varepsilon(a)\langle \varepsilon,f\rangle g(1)}=\langle af\otimes bg,\ 1\otimes 1\rangle$$
$$=\varepsilon(a)\varepsilon(b)f(1)g(1)$$

Similarly, we have $\bar{R}R=1$.

$a' \quad a''$

② $$\Delta^{op}(fa)R=\sum_{i,(f),(a)} f''a''e_i\otimes f'a'e^i=\sum_{i,(f),(a)} f''a'''e_i\otimes f'e^i(S^{-1}(a''')\_\,a')a''$$

$$R(fa)=\sum_{i,(f),(a)} e_if'a'\otimes e^if''a''=\sum_{\substack{i,(f),(a)\\(e_i)}} f'(S^{-1}(e_i''')\_\,e_i')e_i''a'\otimes e^if''a''$$

Evaluate at $bg\otimes ch$.

$$LHS = \sum_{i,(f),(a)} \langle b, f'' \rangle \langle a''''e_i, g \rangle \langle c, \underline{f'}\,\underline{e^i(S^{-1}(a''')\_a')} \rangle \langle a'', h \rangle$$

$$= \sum_{i,(f),(a),(c)} \langle b, f'' \rangle \langle a''''e_i, g \rangle \langle c', f' \rangle \underline{\langle c'', e^i(S^{-1}(a''')\_a') \rangle} \langle a'', h \rangle$$

$$= \sum_{i,(a),(c),(f)} \langle b, f'' \rangle \langle a''''e_i, g \rangle \langle c', f' \rangle \underline{\langle S^{-1}(a''')c''a', e^i \rangle} \langle a'', h \rangle$$

$$= \sum_{i,(a),(c)} \langle bc', f \rangle \langle a'''' \langle S^{-1}(a''')c''a', e^i \rangle e_i, g \rangle \langle a'', h \rangle$$

$$= \sum_{i,(a),(c)} \langle bc', f \rangle \langle a''''S^{-1}(a''')c''a', g \rangle \langle a'', h \rangle \qquad a' \; a'' \; \underbrace{a''' \; a''''}$$

$$= \sum_{i,(a),(c)} \langle bc', f \rangle \langle \varepsilon(a''')c''a', g \rangle \langle a'', h \rangle = \sum_{i,(a),(c)} \langle bc', f \rangle \langle c''a', g \rangle \langle a'', h \rangle.$$

$$RHS = \sum_{i,(f),(a),(e_i)} \langle b, f'(S^{-1}(e_i''')\_e_i') \rangle \langle e_i''a', g \rangle \langle c, e^i f'' \rangle \langle a'', h \rangle$$

$$= \sum_{i,(f),(a),(e_i),(c)} \underline{\langle S^{-1}(e_i''')be_i', f' \rangle} \langle e_i''a', g \rangle \langle c', e^i \rangle \underline{\langle c'', f'' \rangle} \langle a'', h \rangle$$

$$= \sum_{i,(a),(c),(e_i)} \langle c'' S^{-1}(e_i''')be_i', f \rangle \langle e_i''a', g \rangle \langle a'', h \rangle \langle c', e^i \rangle \cdots (1)$$

Recall $c = \sum \langle c, e^i \rangle e_i$. Taking $\Delta^{(2)}$, we have

$$\sum_{(c)} c' \otimes c'' \otimes c''' = \sum_i \langle c, e^i \rangle \; e_i' \otimes e_i'' \otimes e_i'''. \qquad \text{Hence (1) becomes}$$

calculate for c'.

$$RHS = \sum_{(a),(c)} \langle c''''S^{-1}(c''')bc', f \rangle \langle c''a', g \rangle \langle a'', h \rangle$$

$$= \sum_{(a),(c)} \langle bc', f \rangle \langle c''a', g \rangle \langle a'', h \rangle$$

Hence we are done.

③ Now check $(\Delta \otimes \mathrm{id}) R = R_{13} R_{23}$.

$$\text{左} = \sum_{i,(e_i)} e_i' \otimes e_i'' \otimes e^i, \qquad \text{右} = \sum_{i,j} e_i \otimes e_j \otimes e^i e^j .$$

Evaluate at $af \otimes bg \otimes ch$.

$$\mathrm{LHS} = \sum_{i,(e_i)} \langle e_i', f\rangle \langle e_i'', g\rangle \langle c, e^i\rangle \, \varepsilon(a)\varepsilon(b) h(1)$$

calculate for c

Recall $\displaystyle\sum_{(x)} x' \otimes x'' = \Delta(x) = \Delta\Big(\sum_i \langle x, e^i\rangle e_i\Big) = \sum_i \langle x, e^i\rangle e_i' \otimes e_i''$

Hence $\displaystyle\mathrm{LHS} = \sum_{(c)} \langle c', f\rangle \langle c'', g\rangle \, \varepsilon(a)\varepsilon(b) h(1)$ or $\mathrm{LHS} = \sum_i \langle e_i, fg\rangle \sim$ $= \langle c, fg\rangle \sim$

$$= \varepsilon(a)\varepsilon(b) \langle c, fg\rangle h(1).$$

$$\mathrm{RHS} = \sum_{i,j} \langle e_i, f\rangle \langle e_j, g\rangle \langle c, e^i e^j\rangle \, \varepsilon(a)\varepsilon(b) h(1)$$

$$= \sum_{i,j,(c)} \langle e_i, f\rangle \langle e_j, g\rangle \langle c', e^i\rangle \langle c'', e^j\rangle \, \varepsilon(a)\varepsilon(b) h(1)$$

$$= \sum_{(c)} \langle c', f\rangle \langle c'', g\rangle \, \varepsilon(a)\varepsilon(b) h(1)$$

$$= \varepsilon(a)\varepsilon(b) \langle c, fg\rangle h(1).$$

Similarly, we have $(\mathrm{id} \otimes \Delta) R = R_{13} R_{12}$.

<u>Rmk</u> We can proceed as follows: for example

$$\sum_{i,(e_i)} \langle e_i''' b S^{-1}(e_i'), f\rangle \langle e_i'' a, g\rangle \langle c, e^i\rangle = \sum \langle S^{-1}(e_i'), f'\rangle \langle b, f''\rangle \langle e_i''', f'''\rangle \langle e_i'', g''\rangle \langle a, g'\rangle \langle c, e^i\rangle$$

$$= \sum \langle e_i', (S^{-1})^* f'\rangle \langle e_i'', g''\rangle \langle e_i''', f'''\rangle \langle c, e^i\rangle \langle a, g'\rangle \langle b, f''\rangle$$

$$= \sum \langle e_i, ((S^{-1})^* f') g'' f'''\rangle \langle c, e^i\rangle \langle a, g'\rangle \langle b, f''\rangle = \sum \langle c, ((S^{-1})^* f') g'' f'''\rangle \langle a, g'\rangle \langle b, f''\rangle$$

$$= \sum \langle c''' b S^{-1}(c'), f\rangle \langle c'' a, g\rangle$$

*More remark on duals

Let
$$\begin{cases} e_i e_j = \sum_k a_{ij}^k e_k, \\ \Delta(e_i) = \sum_{j,k} b_i^{jk} e_j \otimes e_k. \end{cases}$$

Then $X=(H^{op})^*$ has following structure constants,

$$\begin{cases} \Delta(e^k) = \sum_{i,j} a_{ji}^k e^i \otimes e^j & \leftarrow a_{ij}^k \text{ of } H^*. \\ e^j e^k = \sum_i b_i^{jk} e^i \end{cases}$$

For example, $\Delta(e^k) = \sum_{i,j} c_{ij}^k e^i \otimes e^j$. Then

$$\langle e_p e_q, e^k\rangle = \sum_{(e^k)} \langle e_q, (e^k)'\rangle \langle e_p, (e^k)''\rangle.$$ Hence

$$\underbrace{\sum_r a_{pq}^r \langle e_r, e^k\rangle}_{= a_{pq}^k} = \underbrace{\sum_{i,j} c_{ij}^k \langle e_q, e^i\rangle\langle e_p, e^j\rangle}_{= c_{qp}^k}.$$

On the Hopf algebra,
$$\begin{cases} \sum S(a')a'' = \varepsilon(a)1 = \sum a'S(a'') \\ \sum \varepsilon(a')a'' = a = \sum a'\varepsilon(a'') \end{cases}$$

Δ : multiplicative $\Rightarrow \Delta(ab) = \sum_{(a),(b)} a'b' \otimes a''b''$

S : $\left(\begin{array}{l}\text{anti-multiplicative} \\ (S\otimes S)\Delta = \Delta^{op} S\end{array}\right) \Rightarrow \Delta(S(a)) = \sum_{(a)} S(a'')\otimes S(a')$,

$\varepsilon\circ S = \varepsilon$