On the Relations between Quantum Entanglement and Topological Invariant

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Abstract

This thesis gives a criterion for detecting the entanglement of a quantum state, and uses it to study the relationship between topological and quantum entanglement. It is fundamental to view topological entanglements such as braids entanglement operators and to associate to them unitary operators that are capable of creating quantum entanglement. The entanglement criterion is used to explore this connection [L.H. Kauffman and S.J. Lomonaco, quant-ph/0304091, 2003].

Keywords: quantum entanglement, knot invariant, braiding, Yang-Baxter operator, linking, writhe

1 Introduction

Quantum entanglement is a very astonishing phenomenon in natural science. By using this property, there are many useful results such as quantum dense coding, quantum key distribution, quantum teleportation, quantum information theory, and quantum computation.

This paper discusses relationships between topological entanglement and quantum entanglement. Kauffman and Lomonaco proposed that it is more fundamental to view topological entanglements such as braids as entanglement operators and to associate with them unitary operators that perform quantum entanglement. Then one can compare the way the unitary operator corresponding to an elementary braid has (or has not) the capacity to entangle quantum states. Recently, they are interested in the role of unitary braiding operators in quantum computing; certain solutions to the Yang-Baxter equation form universal gates in the presence of local unitary transformations [2].

In Section 2 we will introduce axioms of quantum mechanics and quantum entanglement. In Section 3 we prove a set of equations that characterize entanglement of an n-qudit quantum state. Section 4 then shows how this criterion can be used to analyze a general class of solutions to the Yang-Baxter equation and their corresponding link invariants just as the paper [1] of Kauffman and Lomonaco in 2003.

2 Preliminaries

2.1 Axioms of Quantum mechanics

Axiom 1 (State). A state is a ray in a Hilbert space over the complex field $\mathbb{C}$ and the ray is an equivalence class of vectors that differ by multiplication by a nonzero complex number $\mathbb{C}$.

We remark that each state is considered as a unit vector in a Hilbert space over $\mathbb{C}$.

In quantum two-level system $\mathbb{C}^2$, $\{|0\rangle, |1\rangle\}$ forms a basis in $\mathbb{C}^2$. A state can be written as $\alpha|0\rangle + \beta|1\rangle$, where $\alpha,\beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$.

In n-qubit register,

$$\{\ |x_{n-1}\rangle \otimes |x_{n-2}\rangle \otimes \cdots \otimes |x_0\rangle \mid x_i = 0 \text{ or } 1 \text{ for } i = 0, 1, \ldots, n-1\}$$

forms a basis of $\underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n}$.

$$|x_{n-1}\rangle \otimes |x_{n-2}\rangle \otimes \cdots \otimes |x_0\rangle \stackrel{\text{def}}{=} |x_{n-1}, x_{n-2}, \ldots, x_0\rangle \stackrel{\text{def}}{=} |x_{n-1}x_{n-2}\cdots x_0\rangle$$

In n-qudit register,

$$\{|x_{n-1}x_{n-2}\cdots x_0\rangle \mid x_i = 0, 1, \ldots, d-1 \text{ for } i = 0, 1, \ldots, n-1\}$$

forms a basis of $(\mathbb{C}^d)^{\otimes n}$.

Axiom 2 (Observable). An observable is a self-adjoint (or, Hermitian) operator on a complex Hilbert space.

A self-adjoint operator in a Hilbert space $\mathcal{H}$ has a spectral representation; its eigenstates form a complete orthonormal basis in $\mathcal{H}$. We can express a self-adjoint operator $\mathbf{A}$ as

$$\mathbf{A} = \sum_n a_n P_n,$$

where each a_n is an eigenvalue of $\mathbf{A}$, and P_n is the corresponding orthogonal projection onto the eigenspace with eigenvalue a_n.

Axiom 3 (Measurement). In quantum mechanics the numerical outcome of a measurement of the observable $\mathbf{A}$ is an eigenvalue of $\mathbf{A}$.

If the quantum state just prior to the measurement is $|\Psi\rangle$, then the outcome a_n is obtained with probability

$$Prob(a_n \text{ is observed}) = ||P_n|\Psi\rangle||^2 = \langle\Psi|P_n|\Psi\rangle.$$

Axiom 4 (Dynamics). The state evolution of a closed quantum system is determined by unitary operators.

In the Schrödinger picture of dynamics, the vector describing the system moves in time as governed by the Schrödinger equation

$$\frac{d}{dt}|\Psi(t)\rangle = -i\mathbf{H}|\Psi(t)\rangle,$$

where $\mathbf{H}$ is the Hamiltonian. We may reexpress this equation, to first order in the infinitesimal quantity dt, as

$$|\Psi(t+dt)\rangle = (1 - i\mathbf{H}dt)|\Psi(t)\rangle.$$

The operator $U(dt) \equiv 1 - i\mathbf{H}dt$ is unitary; because $\mathbf{H}$ is self-adjoint it satisfies $U^\dagger U = 1$ to linear order in dt. Since a product of unitary operators is finite, time evolution over a finite interval is also unitary

$$|\Psi(t)\rangle = U(t)|\Psi(t)\rangle.$$

In the case where $\mathbf{H}$ is t-independent; we may write $U = e^{-it\mathbf{H}}$.

2.2 Quantum Entanglement

Let Q_1 and Q_2 be two quantum systems with their underlying Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be the states of a combined quantum system $Q = Q_1 \otimes Q_2$ given by the tensor product of the states, that is,

$$|\psi_1\rangle \otimes |\psi_2\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2. \tag{1}$$

Definition 2.1. The state is called **unentangled** (or **separable**) if it is of the form in the equation (1). Otherwise, the state is called **entangled**.

Definition 2.2. Let $Q_1, Q_2, \ldots, Q_n$ be quantum systems with underlying Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$ respectively. Then the global quantum system Q_{global} consisting of the quantum systems $Q_1, Q_2, \ldots, Q_n$ is said to be **entangled** if its state $|\Psi\rangle \in \mathcal{H}_{global} = \otimes_{j=1}^{n} \mathcal{H}_j$ can be written in the form

$$|\Psi\rangle = \otimes_{j=1}^{n} |\psi_j\rangle,$$

where each $|\psi_j\rangle$ lies in the Hilbert space $\mathcal{H}_j$ for $j = 1, 2, \ldots, n$. We also say that such a state $|\Psi\rangle$ is **entangled**.

3 Entanglement Criteria

Let us consider a state $|\Psi\rangle \in (\mathbf{C^d})^{\otimes n}$. The state can be written as $|\Psi\rangle = \sum_\alpha a_\alpha |\alpha\rangle$, where

$$\alpha = \underbrace{00\cdots 0}_{n}, \underbrace{00\cdots 1}_{n}, \ldots, \underbrace{d-1\, d-1 \cdots d-1}_{n}$$

$$a_\alpha \in \mathbf{C} \quad \text{and} \quad \sum_\alpha |a_\alpha|^2 = 1.$$

Notation 3.1. Let $|\alpha|$ denote the number of nonzero terms in the string α, e_i the string of length n with all zeros except for a 1 in the i-th place, and $i \in \alpha$ the i-th place in the string α is occupied by nonzeros.

Theorem 3.1. *The state $|\Psi\rangle = \sum_\alpha a_\alpha |\alpha\rangle$ is unentangled if and only if*

$$a_{00\ldots 0}^{|\alpha|-1} a_\alpha = \prod_{i \in \alpha} a_{(\alpha \cdot e_i) e_i}$$

for all α.

Proof. The state $|\Psi\rangle$ rewrites that

$$\begin{aligned}|\Psi\rangle &= a_{00\cdots 0}|00\cdots 0\rangle + a_{10\cdots 0}|10\cdots 0\rangle + \cdots + a_{d-1\,d-1\cdots d-1}|d-1\,d-1\,\cdots d-1\rangle \\ &= a_{00\cdots 0}\left(|00\cdots 0\rangle + \frac{a_{10\cdots 0}}{a_{00\cdots 0}}|10\cdots 0\rangle + \cdots + \frac{a_{d-1\,d-1\cdots d-1}}{a_{00\cdots 0}}|d-1\,d-1\,\cdots d-1\rangle\right).\end{aligned} \tag{2}$$

If $|\Psi\rangle$ is unentangled then $|\Psi\rangle$ has the form of an n-fold tensor product as shown below,

$$\begin{aligned}|\Psi\rangle =& a_{00\cdots 0}(|0\rangle + b_{10\cdots 0}|1\rangle + b_{20\cdots 0}|2\rangle + \cdots + b_{d-1\,0\cdots 0}|d-1\rangle) \\ &\otimes (|0\rangle + b_{01\cdots 0}|1\rangle + b_{02\cdots 0}|2\rangle + \cdots + b_{0\,d-1\cdots 0}|d-1\rangle) \\ &\otimes \cdots \otimes (|0\rangle + b_{00\cdots 1}|1\rangle + b_{00\cdots 2}|2\rangle + \cdots + b_{00\cdots d-1}|d-1\rangle).\end{aligned} \tag{3}$$

It follows from (2) and (3) that

$$b_{(\alpha\cdot e_i)e_i} = \frac{a_{(\alpha\cdot e_i)e_i}}{a_{00\ldots 0}} \ . \tag{4}$$

Also, we obtain from (2), (3), and (4) that

$$a_\alpha = a_{00\cdots 0}\prod_{i\in\alpha}\frac{a_{(\alpha\cdot e_i)e_i}}{a_{00\cdots 0}} \ .$$

$$\text{Hence} \quad a_{00\cdots 0}^{|\alpha|-1}a_\alpha = \prod_{i\in\alpha} a_{(\alpha\cdot e_i)e_i} \quad \text{for all} \quad \alpha.$$

Conversely, if the coefficients of $|\Psi\rangle$ satisfy this formula, then $|\Psi\rangle$ factorizes in the above form. □

Remark 3.1. The simplest example of the theorem is the case of two entangled qubit. By the above theorem,

$$|\Psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle \in (\mathcal{C}^2)^{\otimes 2}$$

is unentangled exactly when $a_{00}a_{11} = a_{01}a_{10}$.
Two entangled qutrit in $(\mathcal{C}^3)^{\otimes 2}$

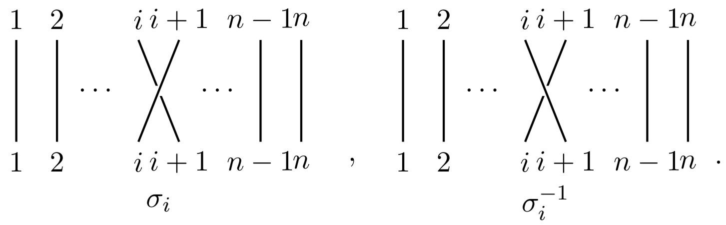
$$\begin{aligned}|\Psi\rangle =& a_{00}|00\rangle + a_{01}|01\rangle + a_{02}|02\rangle + a_{10}|10\rangle + a_{11}|11\rangle + a_{12}|12\rangle + a_{20}|20\rangle \\ &+ a_{21}|21\rangle + a_{22}|22\rangle\end{aligned}$$

is unentangled exactly when

$$\begin{aligned} a_{00}a_{11} =& a_{10}a_{01}, \\ a_{00}a_{12} =& a_{10}a_{02}, \\ a_{00}a_{21} =& a_{20}a_{01}, \\ a_{00}a_{22} =& a_{20}a_{02}. \end{aligned}$$

4 Topological Invariant

We define σ_i as the braid where the i^{th} strand goes under and the right of the $i+1^{th}$ and $\sigma_i{}^{-1}$ the braid where the i^{th} strand goes over and the right of the $i+1^{th}$,
that is,



For any $n \geq 1$ the n-braid group $\mathbf{B}_n$ has the following presentation,

$$\mathbf{B}_n = \left\langle \sigma_1, \sigma_2, \cdots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } i = 1, 2, \ldots, n-2 \\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2 \end{array} \right\rangle.$$

We now try to obtain representation $\phi_n : \mathbf{B}_n \to End(V^{\otimes n})$ defined by

$$\phi_n(\sigma_i) = (id_V)^{\otimes(i-1)} \otimes R \otimes (id_V)^{\otimes(n-i-1)} \tag{5}$$

for some invertible linear map $R : V \otimes V \to V \otimes V$. To obtain such a representation ϕ_n of the braid group $\mathbf{B}_n$ from the map R, the map ϕ_n is required to satisfy the following relations,

$$\phi_n(\sigma_i\sigma_j) = \phi_n(\sigma_j\sigma_i) \text{ for } |i-j| \geq 2 \tag{6}$$

and

$$\phi_n(\sigma_i\sigma_{i+1}\sigma_i) = \phi_n(\sigma_{i+1}\sigma_i\sigma_{i+1}) \text{ for } i = 1, 2, \ldots, n-2. \tag{7}$$

Let us consider what is required of a map R so that ϕ_n will satisfy the above relations, (6) and (7). Such a given map ϕ_n always satisfies (6).
To obtain (7), the linear map R is required to satisfy the relation

$$(R \otimes id_V)(id_V \otimes R)(R \otimes id_V) = (id_V \otimes R)(R \otimes id_V)(id_V \otimes R).$$

We call this equation **the Yang-Baxter equation**, and the matrix associated with the linear map R an R **matrix**. For the invertible linear map R we obtain a representation ϕ_n of the braid group $\mathbf{B}_n$ by (5).

Let $\{v_0, v_1, \cdots, v_{d-1}\}$ be a basis of V and R a matrix

$$\mathbf{R} = \begin{pmatrix} \cdots & R_{i\,j}^{0\,0} & \cdots \\ \cdots & R_{i\,j}^{0\,1} & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & R_{i\,j}^{k\,l} & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & R_{i\,j}^{d-1\,d-1} & \cdots \end{pmatrix}.$$

Then we have

$$R(v_i \otimes v_j) = \sum_{k,l} R_{i\,j}^{k\,l} v_k \otimes v_l,$$

that is,

$$\begin{pmatrix} \cdots & R_{i\,j}^{0\,0} & \cdots \\ \cdots & R_{i\,j}^{0\,1} & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & R_{i\,j}^{k\,l} & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & R_{i\,j}^{d-1\,d-1} & \cdots \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ a_{i\,j} \\ \vdots \\ 0 \end{pmatrix} = a_{i\,j} \begin{pmatrix} R_{i\,j}^{0\,0} \\ R_{i\,j}^{0\,1} \\ \vdots \\ R_{i\,j}^{k\,l} \\ \vdots \\ R_{i\,j}^{d-1\,d-1} \end{pmatrix}.$$

We change the Yang-Baxter equation into the equation of component of R matrix;

$$\begin{aligned}
(L.H.S.) &= (R\otimes id_V)(id_V\otimes R)(R\otimes id_V)(v_i\otimes v_j\otimes v_k)\\
&= (R\otimes id_V)(id_V\otimes R)(\sum_{u,w} R^{u\,w}_{i\,j} v_u\otimes v_w\otimes v_k)\\
&= (R\otimes id_V)(\sum_{u,w,p,q} R^{u\,w}_{i\,j}R^{p\,q}_{w\,k} v_u\otimes v_p\otimes v_q)\\
&= \sum_{u,w,p,q,r,s} R^{u\,w}_{i\,j}R^{p\,q}_{w\,k}R^{r\,s}_{u\,p} v_r\otimes v_s\otimes v_q
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
(R.H.S.) &= (id_V\otimes R)(R\otimes id_V)(id_V\otimes R)(v_i\otimes v_j\otimes v_k)\\
&= (id_V\otimes R)(R\otimes id_V)(\sum_{x,y} R^{x\,y}_{j\,k}(v_i\otimes v_x\otimes v_y))\\
&= (id_V\otimes R)\sum_{x,y,r,z} R^{x\,y}_{j\,k}R^{r\,z}_{i\,x}(v_r\otimes v_z\otimes v_y)\\
&= \sum_{x,y,r,z,s,q} R^{x\,y}_{j\,k}R^{r\,z}_{i\,x}R^{s\,q}_{z\,y} v_r\otimes v_s\otimes v_q.
\end{aligned} \tag{9}$$

By (8) and (9) we have

$$\sum_{u,w,p} R^{u\,w}_{i\,j}R^{p\,q}_{w\,k}R^{r\,s}_{u\,p} = \sum_{x,y,z} R^{x\,y}_{j\,k}R^{r\,z}_{i\,x}R^{s\,q}_{z\,y} \tag{10}$$

for all choices of i, j, k and q, r, s.

Let $M=(M_{ab})$ be an $d\times d$ matrix whose entries $M_{ab}\in\mathcal{C}$ with $|\,M_{ab}|=1$. Define an R matrix by

$$R^{a\,b}_{c\,d} = M_{ab}\delta^a_d\delta^b_c. \tag{11}$$

Then

$$R(v_i\otimes v_j) = M_{ji}\, v_j\otimes v_i.$$

By (10) and (11) it then follows that

$$(L.H.S.) = M_{j\,i}M_{k\,i}M_{k\,j} = M_{k\,j}M_{k\,i}M_{j\,i} = (R.H.S.)$$

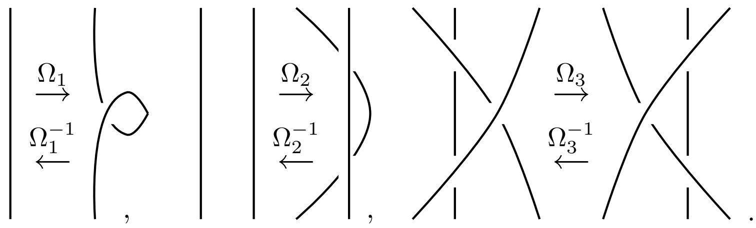
and

$$RR^* = I_{d^2},$$

where R^* is the adjoint matrix of R, and hence R is a unitary solution of Yang-Baxter Equation [3, 4].

In this part we shall define knot Reidemeister moves, linking number, writhe, and state summation.

Reidemeister discovered a simple set of moves on link diagrams that captures the concept of ambient isotopy of knots in three dimensional space. There are three basic Reidemeister moves. Reidemeister's theorem states that two diagrams represent ambient isotopic knots (or links) if and only if there is a sequence of Reidemeister moves taking one diagram to the other. The Reidemeister moves are illustrated as below.



We want to be able to calculate numbers(or bits of algebra such as polynomials) from given link diagrams in such a way that these numbers do not change when the diagrams are changed by Reidemeister moves. Numbers or polynomials of this kind are called **invariants** of the knot or link represented by the diagram. If we produce such invariants, then we find topological information about the knot or link.
If we can change a regular diagram, $\mathbf{D}$, to another $\mathbf{D}'$ by performing, a finite number of times, the operations $\Omega_1, \Omega_2, \Omega_3$ and/or their inverses, then $\mathbf{D}$ and $\mathbf{D}'$ are said to be **equivalent**. We shall denote this equivalence by $\mathbf{D} \approx \mathbf{D}'$.

Theorem 4.1. *Suppose that* $\mathbf{D}$ *and* $\mathbf{D}'$ *are regular diagrams of two knots(or links)* K *and* K'*, respectively. Then*

$$K \approx K' \Leftrightarrow \mathbf{D} \approx \mathbf{D}'.$$

We may conclude, from the above theorem, that the problem of equivalence of knots, in essence, is just a problem of the equivalence of regular diagrams.

Therefore, a knot (or link) invariant may be thought of as a quantity that remain unchanged when we apply any one of the above Reidemeister moves to a regular diagram.

At a crossing point c of an oriented regular diagram, as shown in Figure 1, we have two possible configurations. In case (a) we assign $\mathbf{sign(c)} = \mathbf{+1}$ to the crossing point, while in case (b) we assign $\mathbf{sign(c)} = \mathbf{-1}$. The crossing point in (a) is said to be positive, while that in (b) is said to be negative.

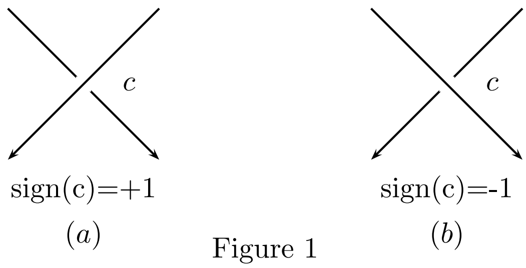


Figure 1

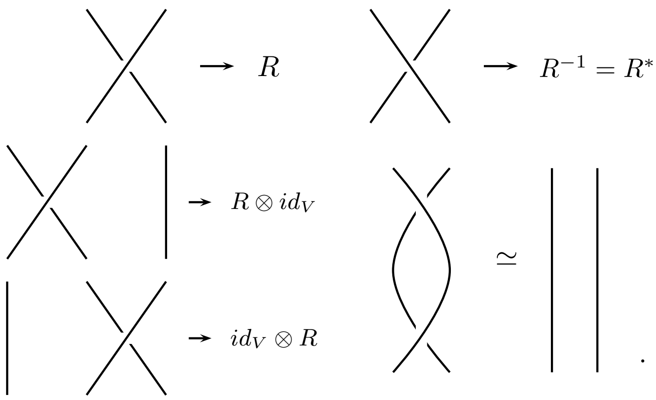
Suppose that $\mathbf{D}$ is an oriented regular diagram of a 2-component link $K = \{K_1, K_2\}$. Further, suppose that the crossing points $\mathbf{D}$ at which the projections of K_1 and K_2 intersect are $c_1, c_2, \ldots, c_m$. (We ignore the crossing points of the projections of K_1 and K_2, which are self-intersections of the knot component.) Then

$$\frac{1}{2}\{sign(c_1) + sign(c_2) + \cdots + sign(c_m)\}$$

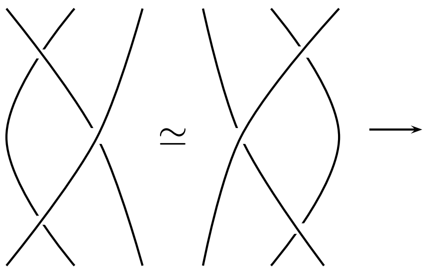
is called the **linking number** of K_1 and K_2, which we will denote by $lk(K_1, K_2)$. Suppose that $\mathbf{D}$ is an oriented regular diagram of an oriented knot (or link). Then, the sum $w(\mathbf{D})$ of the signs of all the crossing points of $\mathbf{D}$ is said to be the **writhe** of $\mathbf{D}$.

Theorem 4.2. *The writhe of an oriented regular diagram is invariant under the Reidemeister moves* Ω_2, Ω_3, *and their inverses.*

Now we can associate a unitary operator to an elementary braid as follows [2, 3].



Then we can view the relation between first braid relation and the Yang-Baxter equation as follows.



$$(R \otimes id_V)(id_V \otimes R)(R \otimes id_V)$$
$$\|$$
$$(id_V \otimes R)(R \otimes id_V)(id_V \otimes R)$$

The Yang-Baxter equation .

Suppose γ is a n-braid and $\mathbf{D}$ is a regular diagram of γ. At each crossing point of $\mathbf{D}$, let us look at the four segments that make up a neighborhood of that crossing point. We may assign the basis states in $(\mathbb{C}^d)^n$,

$$|\underbrace{00\cdots 0}_{n}\rangle, |\underbrace{00\cdots 1}_{n}\rangle, \cdots, |\underbrace{d-1\ d-1\cdots d-1}_{n}\rangle,$$

on the braid by placing a state on each of these four segments at each crossing point. For this given state, we may assign a Boltzmann weight at each crossing point of $\mathbf{D}$, as described below. On the four segments close to a crossing point suppose the state are assigned as shown in the Figure 2.

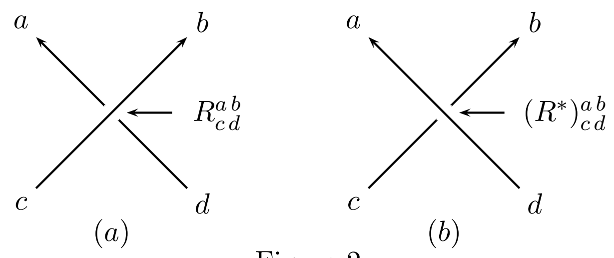


Figure 2

Then, if the crossing point is positive, Figure 2(a), then assign $R^{a\,b}_{c\,d}$ $(= M_{ab}\delta^a_d\delta^b_c)$ to the crossing point; if the crossing point is negative, Figure 2(b), then assign $(R^*)^{a\,b}_{c\,d}$ to the crossing point. We form a knot(or link) from a braid by adding closure strings, Figure 3.

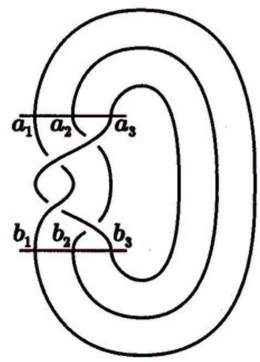


Figure 3

These closure strings will also have a contribution to a subsequent knot invariant. Hence we need also assign state variables to these closure strings. But if, for a given state, a_k is assigned to the top half of the k^{th} closure string, and the state b_k to the bottom half of the k^{th} closure string, then we shall assume they are equal, see Figure 3.

Corresponding to an oriented link diagram K, we define the state summation S_K by summing over all assignments of strings α to each component of the link K, (colorings of the diagram K) and taking the product of the matrix entries $M_{\alpha,\,\beta}$ associated via R to each crossing in the colored diagram [2, 3].

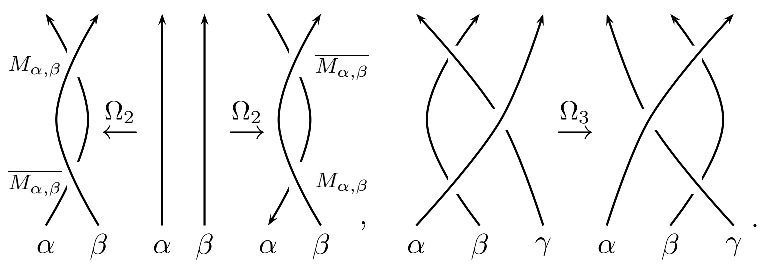
If we consider that if K is a link of two components K_1 and K_2, then

$$S_K = \sum_{\alpha \neq \beta} M_{\alpha,\alpha}^{w(K_1)} M_{\beta,\beta}^{w(K_2)} M_{\alpha,\beta}^{2lk(K_1,K_2)} + \sum_{\alpha} M_{\alpha,\alpha}^{w(K_1)} M_{\alpha,\alpha}^{w(K_2)} M_{\alpha,\alpha}^{2lk(K_1,K_2)}.$$

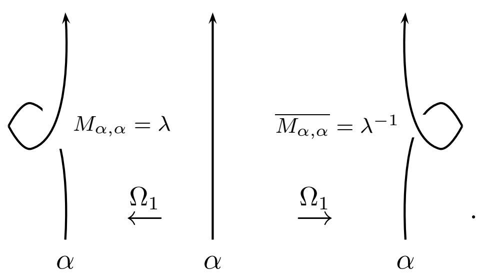
In order to separate out the topological dependence so that we can see how this state summation can detect the linking number of the link, it is useful to assume that $M_{\alpha\,\alpha} = \lambda$ is a constant independent of the string α. We can then write the formula for the state sum in the form

$$\begin{aligned} S_K &= \sum_{\alpha \neq \beta} \lambda^{w(K_1)} \lambda^{w(K_2)} M_{\alpha,\beta}^{2lk(K_1,K_2)} + \sum_{\alpha} \lambda^{w(K_1)+w(K_2)+2lk(K_1,K_2)} \\ &= \sum_{\alpha \neq \beta} \lambda^{w(K)} (\frac{M_{\alpha,\beta}^2}{\lambda^2})^{lk(K_1,K_2)} + \sum_{\alpha} \lambda^{w(K)} \\ &= \lambda^{w(K)} (\sum_{\alpha \neq \beta} (\frac{M_{\alpha,\beta}^2}{\lambda^2})^{lk(K_1,K_2)} + d^n). \end{aligned}$$

By our R matrix of the form (11), S_K should immediately bring to mind the Reidemeister move Ω_2. Moreover, the Yang-Baxter equation is essentially nothing but the Reidemeister move Ω_3 as follows.



However, S_K is not invariant under the first Reidemeister move Ω_1 as follows.



Therefore, by Kauffman's principle it should yield a knot invariant by a multiplication S_K by $\lambda^{-w(K)}$; $\lambda^{-w(K)}S_K$ is invariant under $\Omega_1, \Omega_2, \Omega_3$ and their inverses by Theorem 4.2.
Thus we obtain the topological invariant Z_K defined by the equation

$$Z_K = \lambda^{-w(K)} S_K = \sum_{\alpha \neq \beta} (\frac{M^2_{\alpha,\beta}}{\lambda^2})^{lk(K_1,K_2)} + d^n.$$

We conclude that Z_K can detect linking number so long as $M^2_{\alpha,\beta} \neq \lambda^2$.

Now lets return the matrix R and see about its entanglement capabilities. We are assuming that all the $M_{\alpha,\alpha}$ are equal to λ.
Then if the unentangled $2n$-qudit state $|\Phi\rangle = \frac{1}{d^n} \sum_{\alpha,\,\beta} |\alpha, \beta\rangle$, then

$$R|\Phi\rangle = \frac{1}{d^n} \sum_{\alpha,\,\beta} M_{\alpha,\beta} |\alpha, \beta\rangle.$$

Using our theorem 3.1 and writing 0 for the zero string $\underbrace{00\cdots0}_{n}$, we conclude that the state $R|\Phi\rangle$ is unentangled exactly when the following equations are satisfied for all α and β.

$$\lambda^{|\alpha|+|\beta|-1} M_{\alpha,\beta} = \prod_{i\in\alpha} M_{(\alpha\cdot e_i)e_i,0} \prod_{j\in\beta} M_{0,(\beta\cdot e_j)e_j}.$$

In the case $\alpha = \beta$ this equation becomes

$$\begin{aligned} \lambda^{|\alpha|+|\alpha|-1} M_{\alpha,\alpha} &= \prod_{i\in\alpha} M_{(\alpha\cdot e_i)e_i,0} \prod_{j\in\alpha} M_{0,(\alpha\cdot e_j)e_j} \\ \lambda^{2|\alpha|} &= \prod_{i\in\alpha} M_{(\alpha\cdot e_i)e_i,0} \prod_{j\in\alpha} M_{0,(\alpha\cdot e_j)e_j}. \end{aligned}$$

Thus, letting

$$m_{\alpha,0} = \prod_{i\in\alpha} M_{(\alpha\cdot e_i)e_i,0}$$

and

$$m_{0,\alpha} = \prod_{j\in\alpha} M_{0,(\alpha\cdot e_j)e_j}$$

we have

$$\lambda^{2|\alpha|} = m_{\alpha,0} m_{0,\alpha}$$

and

$$\lambda^{|\alpha|+|\beta|-1} M_{\alpha,\beta} = m_{\alpha,0} m_{0,\beta}.$$

From these formulas we find that

$$m_{0,\alpha} m_{\alpha,0} m_{0,\beta} m_{\beta,0} \lambda^{-2} M_{\alpha,\beta}^2 = m_{\alpha,0} m_{0,\beta} m_{\alpha,0} m_{0,\beta}.$$

Hence

$$m_{0,\alpha} m_{\beta,0} \lambda^{-2} M_{\alpha,\beta}^2 = m_{0,\beta} m_{\alpha,0}.$$

Therefore

$$\frac{M_{\alpha,\beta}^2}{\lambda^2} = \frac{m_{\alpha,0}}{m_{0,\alpha}} \frac{m_{0,\beta}}{m_{\beta,0}}.$$

The state $R|\Phi\rangle$ is unentangled exactly when this last equation is satisfied. We see from this that if the matrix M is symmetric, then the invariant Z_K detects linking exactly when $R|\Phi\rangle$ is an entangled state. On the other hand, if M is not symmetric, then the invariant can detect linking even when the state $R|\Phi\rangle$ is unentangled.

5 Conclusion

We see that for this specialization of the R matrix of the form (11), the operator R entangles quantum states exactly when if can detect linking numbers in the topological context.

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