Introduction to YBE (SNU. Math.dep.)

Ref
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deVega
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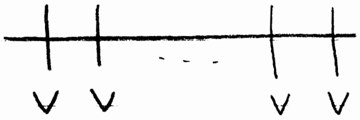
~~Introduction~~
2. 8 vertex & (R) SOS model
3. Algebraic Bethe ansatz
4 finite size corrections & CFT
5 Continuum limit → Quantum Field theory
6 Quantum groups, other models
7. Further Developments : Boundary YBE, 3D Zamolodchikov Eq.

Applications (Overview)

*① quantum spin chains : 1D

(ex) $H : V^{\otimes N} \longrightarrow V^{\otimes N}$

$$H = \sum_{i=1}^{N} \left[\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \cos\gamma \, \sigma_i^z \sigma_{i+1}^z \right]$$



problem : diagonalize H ? → energy levels.

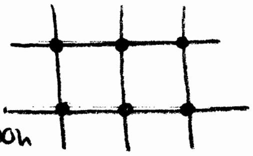
*② quantum field theories ; (1+1) space-time

(ex) sine-Gordon model

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + 2\mu \cos\beta\phi$$

*③ classical stat mechanics on 2D lattice

$\sum_{\text{fluctuation}}$



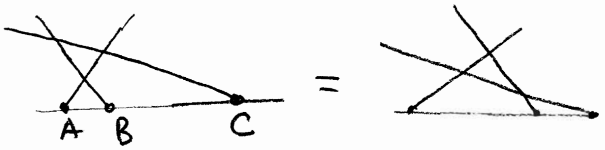
YBE appears when we can solve these EXACTLY

$\Rightarrow$ Integrable or Quantum Integrable models

④ Statistics Matrix

$$\phi_A(x)\,\phi_B(x') = \phi_D(x')\,\phi_C(x)\,S^{CD}_{AB}(x-x')$$

operator



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⑤ Conformal field theory

0. YBE

$\boxed{N^2 \times N^2 \text{ matrix}}$

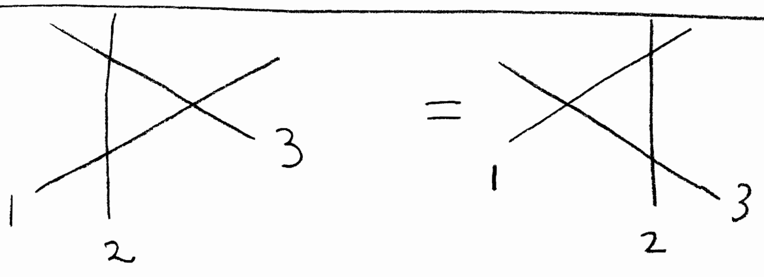
N-dim vector

define $R(u): V \otimes V \to V \otimes V$, $u \in \mathbb{C}$

spectral parameter $R_{12}(u): V^{\otimes 3} \to V^{\otimes 3}$

$= R \otimes \mathbb{1}$

$$\boxed{R_{12}(u)\,R_{13}(u+v)\,R_{23}(v) = R_{23}(v)\,R_{13}(u+v)\,R_{12}(u)}$$



$\rightarrow$ N^6 equations for N^4 unknowns

1. <u>special case</u> $u, v \to 0$

exchange operator $P : x \otimes y \to y \otimes x$

satisfies YBE

$$\therefore \quad \boxed{R(0) = \overset{\text{const}}{c} \cdot P}$$

2. $V = \mathbb{C}^2$; 2dim. $R = 4 \times 4$ matrix

[explicit ex.]

① "Rational" Yang, McGuire

$$R(u) = \begin{pmatrix} 1+u & & & \\ & u & 1 & \\ & 1 & u & \\ & & & 1+u \end{pmatrix} = P + u \cdot \mathbb{1}$$

② "trigonometric" → "quantum group"

$$R(u) = \begin{pmatrix} \sin(\eta+u) & & & \\ & \sin(u) & \sin\eta & \\ & \sin\eta & \sin u & \\ & & & \sin(\eta+u) \end{pmatrix}$$

with η arbitrary const

③ "elliptic" (Baxter)

$$R(u) = \begin{pmatrix} a(u) & & & d(u) \\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ d(u) & & & a(u) \end{pmatrix}$$

$$\begin{array}{llll} a = & \theta_0(\eta) & \theta_0(u) & \theta_1(\eta+u) \\ b \equiv & 0 & 1 & 0 \\ c \equiv & 1 & 0 & 0 \\ d \equiv & 1 & 1 & 1 \end{array}$$

$$\left\{ \begin{array}{l} \theta_0(u, \overset{\text{elliptic modulus}}{p}) = \prod_{n=1}^{\infty} \left(1 - 2p^{n-\frac{1}{2}} \cos 2\pi u + p^{2n-1}\right)(1-p^n) \\ \theta_1(u,p) = 2p^{1/8} \sin \pi u \prod \left(1 - 2p^n \cos 2\pi u + p^{2n}\right)(1-p^n) \end{array} \right.$$

as $p \to 0$ $\quad a, b, c \sim p^{1/8}$ $\quad \boxed{d \to 0}$

$\qquad\qquad\quad d \sim p^{3/8}$ $\quad \to$ trigonometric

$$\boxed{\begin{tikzcd} ① & ② \arrow[l, "\eta \to 0"'] & ③ \arrow[l, "p \to 0"'] \end{tikzcd}}$$

3. Braid relations

define $\quad \check{R}(u) = P\, R(u)$

and $\quad \check{R}_i(u) = \underset{1}{\mathbb{1}} \otimes \cdots \otimes \underset{i,\, i+1}{\check{R}(u)} \otimes \cdots \otimes \underset{N}{\mathbb{1}}$

$R : V_1 \otimes V_2 \to V_2 \otimes V_1$, $\quad \check{R} : V_1 \otimes V_2 \to V_1 \otimes V_2$

$\therefore \check{R}_i : V^{\otimes N} \to V^{\otimes N}$

<u>YBE</u> $\quad \check{R}_{i+1}(u)\, \check{R}_i(u+v)\, \check{R}_{i+1}(v) = \check{R}_i(v)\, \check{R}_{i+1}(u+v)\, \check{R}_1(u)$

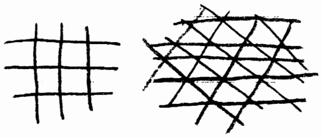
2D Statistical Lattice model

Physics

- with T → system fluctuates

  Boltzman Law of distribution : Prob $\propto e^{-\frac{E_{conf.}}{kT}}$

- Lattice ; many solid material forms well defined lattice
  periodic



on which dynamics are localized on each point.

(ex) ferromagnetism : ↑ or ↓

- interaction ; nearest neighbor only : $\sim \frac{1}{r^5}$: reasonable

(ex) $\boxed{\text{Ising}}$ $\boxed{\sigma = \pm 1}$

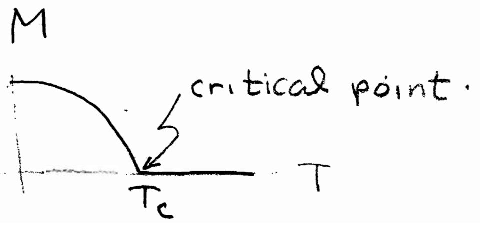
$$H = -J \sum_{\vec{i}} \sigma_{\vec{i}} \, \sigma_{\vec{i}+1} \rightarrow \text{prefers } \uparrow\uparrow \text{ than } \downarrow\uparrow$$

- ground state (minimum energy) → … ↑ ↑ ↑ … / ↑ ↑ ↑ … or $\underbrace{\uparrow \rightarrow \downarrow}_{Z_2 \text{ sym.}}$
  (T→0)
- Thermal fluctuation : other configurations are also possible with less probability
- Partition function :

$$Z = \sum_{conf.} e^{-\beta E}$$

- Due to N→∞, new phenomenon (phase transition) happens

M = ⟨σ⟩ : average spin
magnetization



2D Reformulation (two ways)

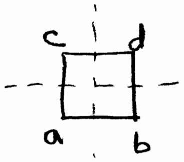
① consider square lattice

1) impose a probability factor on each face □ ⇒ face model

2) on each vertex ┼ ⇒ vertex model

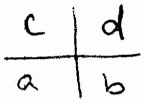
("universality" physics is irrelevant with microscopic details)

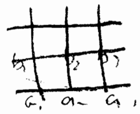
Face



$\rightarrow W\begin{pmatrix} c & d \\ a & b \end{pmatrix}$ "probability"

or



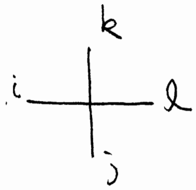


$$Z = \sum_{\{a_i, b_i \cdots\}} \prod_{\text{each } \square} W$$

(ex) Ising : $a = \pm 1$

Boltzman Weight

Vertex



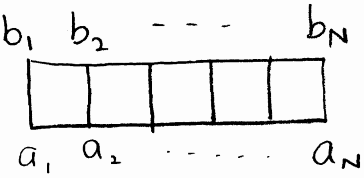
W^{kl}_{ij}

② 1D spin chain

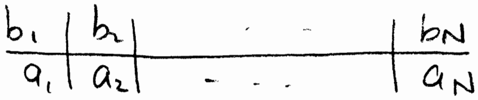
Rewrite

$$Z = \sum_{\{a\}} \sum_{\{b\}} \cdots \cdots T^{\{c\}}_{\{b\}} T^{\{b\}}_{\{a\}}$$

with $T^{\{b\}}_{\{a\}} \equiv$

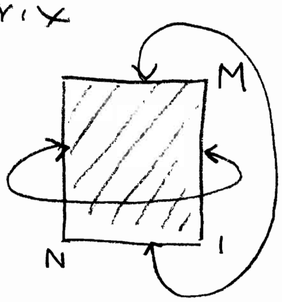


$= W^{b_1 b_2}_{a_1 a_2} W^{b_2 b_3}_{a_2 a_3} \cdots W^{b_{N-1} b_N}_{a_{N-1} a_N}$



T = transfer matrix

PBC of 2D lattice



$$\boxed{Z = \mathrm{Tr}\, T^M}$$

Basic Question of 2D Model

which models have exactly solvable Z

or

or diagonalizable T

To have exact Z , what should be W ?

"Integrable Models"

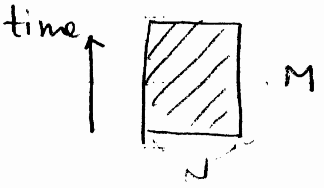
Physicists' answer

- iff there are $\boxed{\text{many}}$ conserved quantities

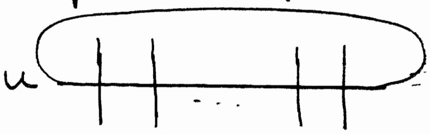
  $\downarrow$ at least # of d.o.f. $\qquad \uparrow$ $\frac{dQ}{dt} = 0$

- $$\frac{dQ}{dt} = [H, Q] = 0 \longrightarrow \underline{[H, Q] = 0}$$

in 2D lattice model



introduce spectral parameter u in T

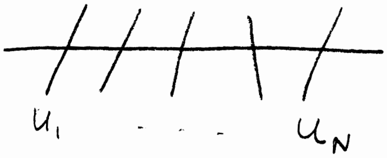
$T(u) = u$ 

commuting Transfer matrix $\quad [T(u), T(v)] = 0$

guarantees infinite # of conserved quantities

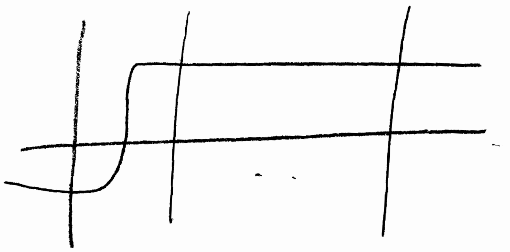
$$T(u) = \sum_{n=0}^{\infty} T_n u^n \longrightarrow [T_n, T_m] = 0$$

furthermore, we introduce inhomogeneity

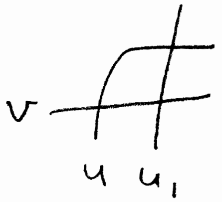


$$= T(u|\{u_i\}) = \prod_{i=1}^{N} W(u-u_i)$$

$=$



if



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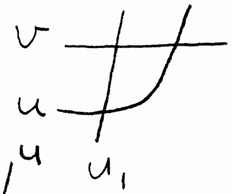
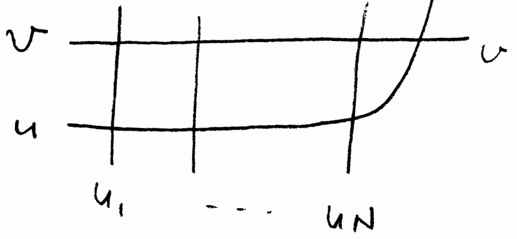
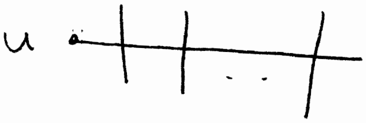


$= \cdots =$



$$R_{12}(u-v)\, T_1(u) \otimes T_2(v) = T_2(v) \otimes T_1(u)\, R_{12}(u-v)$$



$= T(u) : V \to V$ auxiliary space

$V^{\otimes N} \to V^{\otimes N}$ quantum space

$$\mathrm{Tr}_{V\otimes V}\, T_1(u) \otimes T_2(v) = T(u)\, T(v)$$

$$= \mathrm{Tr}_{V\otimes V}\left[R_{12}^{-1}\, T_2 \otimes T_1\, R_{12} \right] = T(v)\, T(u)$$ ✓

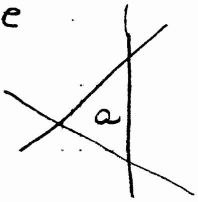
∴ if Boltzman weight satisfies YBE,

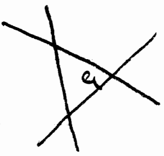
we have <u>integrable models</u>.

"Actual solution to find Z is a different issue!"

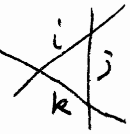
YBE is a sufficient condition for integrability.

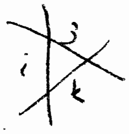
face type

$\sum_a$ 

$= \sum_a$ 

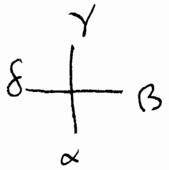
vertex type

$\sum_{ijk}$ 

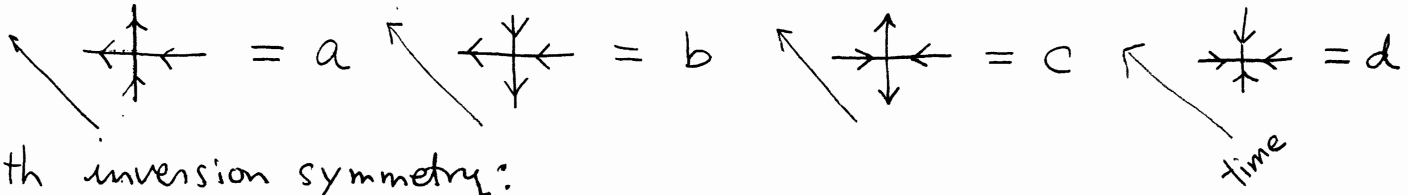
$= \sum_{ijk}$ 

Example

8 Vertex model



$\equiv S^{\gamma\delta}_{\alpha\beta}$



with inversion symmetry:

$$\frac{\theta_1}{\theta_0} = \mathrm{sn}\, u \text{ etc}$$

$$\begin{cases} a = \mathrm{sn}(\gamma - u) \\ b = \mathrm{sn}\, u \\ c = \mathrm{sn}\, \gamma \\ d = p \cdot \mathrm{sn}\gamma\, \mathrm{sn}(\gamma - u) \end{cases}$$ *

as elliptic modulus $p \to 0$ sn → sin 8 vertex → $\boxed{\text{6 vertex}}$

SOS model

$$\begin{array}{c|c} i & k \\ \hline j & l \end{array} = R(i, j; k, l | u)$$

with $|i - j| = 1$ etc.

(ex) $\begin{array}{c|c} l & l-1 \\ \hline l+1 & l \end{array}$, $-\infty < l < \infty$

$$\begin{array}{c|c} l-1 & l \\ \hline l & l+1 \end{array} = \begin{array}{c|c} l+1 & l \\ \hline l & l-1 \end{array} = h(u+\gamma)$$

$$\begin{array}{c|c} l & l\pm1 \\ \hline l\mp1 & l \end{array} = h(u-\gamma)$$

$$\begin{array}{c|c} l & l\pm1 \\ \hline l\pm1 & l \end{array} = \cdots$$

$$\gamma = \frac{2K}{\nu} \quad \nu = 3, 4, \cdots$$

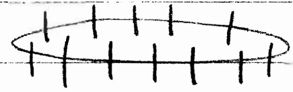
K ↙ elliptic modulus

Relation 1

XXZ spin chain ⟷ 6 Vertex model.

define $H = \frac{\partial}{\partial u} \log T_{6V}(u) \Big|_{u=0}$

"Hamiltonian"



$$\boxed{H_{XXZ} = \sum_{i=1}^{N} \left[\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \cos\gamma \, \sigma_i^z \sigma_{i+1}^z \right]}$$

with PBC $\sigma_1^a = \sigma_{N+1}^a$, $a = x, y, z$

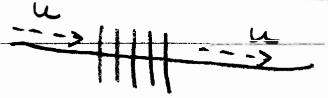
$[T(u), T(v)] = 0 \quad \longrightarrow \quad [H, Q_N] = 0$, $N = 1, \cdots, \infty$

$$R(u) = \begin{pmatrix} \sin(\gamma-u) & & & \\ & \sin u & \sin\gamma & \\ & \sin\gamma & \sin u & \\ & & & \sin(\gamma-u) \end{pmatrix}$$

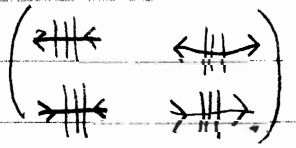
How to diagonalize?

Use YB algebra $\boxed{R_{12}(u-v)\, \tau_1(u) \otimes \tau_2(v) = \tau_2(v) \otimes \tau_1(u) R_{12}}$

- monodromy matrix $\tau(u) =$

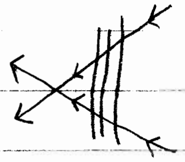


$T(u) = \mathrm{tr}\,\tau = A(u) + D(u)$ $= \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} =$

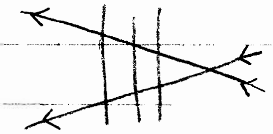


$Z = \mathrm{tr}\, T^M$

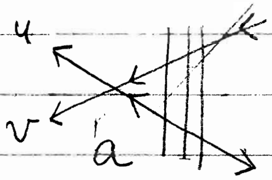
- **YB algebra**



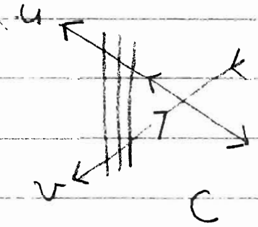
$$b(u-v)\, A(u)\, A(v) = A(v)\, A(u)\, b(u-v)$$



$$\therefore \quad [A(u), A(v)] = 0 = [B(u), B(v)]$$



$$a(u-v)\, B(u)\, A(v) = A(v)\, B(u)\, b(u-v) + c(u-v)\, B(v)\, A(u)$$



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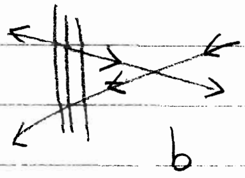
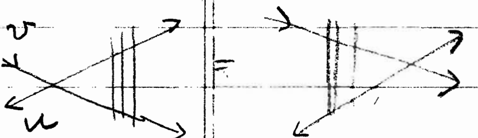
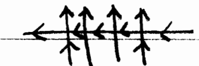


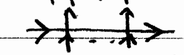
$$\rightarrow \quad \boxed{A(v)\, B(u) = \frac{a}{b}(u-v)\, B(u)\, A(v) - \frac{c}{b}(u-v)\, B(v)\, A(u)}$$



$$\boxed{D(v)\, B(u) = \frac{a}{b}(v-u)\, B(u) D(v) + \frac{c}{b}(u-v)\, B(v) D(u)}$$

(Algebraic) Bethe ansatz

$A|0\rangle =$  $= a(u)^N |0\rangle$

$D|0\rangle =$  $= b(u)^N |0\rangle$

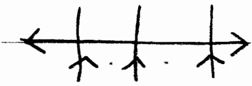
- "pseudo-vacuum" : $\uparrow \uparrow \cdots \uparrow \equiv$  $\equiv |0\rangle$

$C(u)|0\rangle =$  $=$  $= 0$

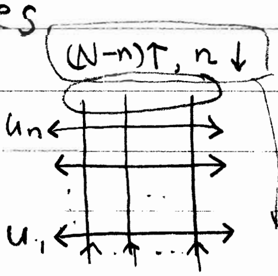
↑ "annihilation operator"

- physical eigenstates

"one ↓", $(N-1)\uparrow$

$B|0\rangle =$ 

$|\Psi\rangle \equiv B(u_1) \cdots B(u_n)|0\rangle$



$= |\downarrow \uparrow \cdots \uparrow\rangle + |\uparrow \downarrow \uparrow \cdots \uparrow\rangle + \cdots$

${}_N C_n$ terms.

and claim that $T(u)|\Psi\rangle = \Lambda(u|\{u_i\})|\Psi\rangle$

(ex)

$$A(u)|\Psi\rangle = A(u)B(u_1)\cdots B(u_n)|0\rangle$$

$$= \frac{a}{b}(u_1-u)\,B(u_1)A(u)B(u_2)\cdots|0\rangle - \frac{c}{b}(u_1-u)\,B(u)A(u_1)B(u_2)\cdots|0\rangle$$

$$= \underbrace{\left[\prod_{j=1}^{n}\frac{a}{b}(u_j-u)\right]\cdot a^N(u)}_{\lambda_A}|\Psi\rangle - \underline{B(u)\left[B(\hat{u}_1)\cdots B|0\rangle\right]}$$

$$D(u)|\Psi\rangle = \underbrace{\left[\prod_{j=1}^{n}\frac{a}{b}(u-u_j)\right]b^N(u)}_{\lambda_D}|\Psi\rangle + \cdots$$

$$T(u)|\Psi\rangle = \Lambda(u)|\Psi\rangle + \underline{\underline{\text{unwanted terms}}}$$

can show that unwanted terms cancel each other

if $$\underline{b(u_j)^N\prod_{i=1}^{n}\frac{a}{b}(u_j-u_i) + a(u_j)^N\prod_{i=1}^{n}\frac{a}{b}(u_i-u_j) = 0}$$

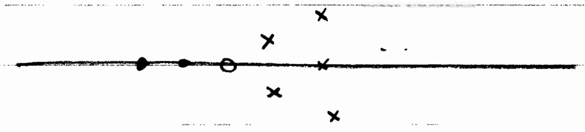
for all j

or $$\boxed{\left[\frac{b}{a}(u_j)\right]^N\prod_{i=1}^{n}\frac{a(u_j-u_i)\,b(u_i-u_j)}{b(u_j-u_i)\,a(u_i-u_j)} = -1}$$

BAE

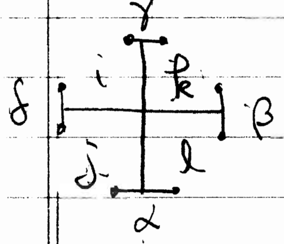
Solution of theses set of algebraic eq.

$\boxed{\text{string ansatz}}$



* <u>8 Vertex</u> above method does not work!

<u>Baxter</u> introduced SOS model which is equivalent to 8V.

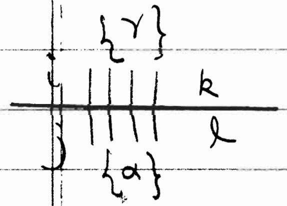


$$\sum_{\alpha\beta} S^{\gamma\delta}_{\alpha\beta}\, X^{\alpha}(j,l)\, Y^{\beta}(k,l)$$

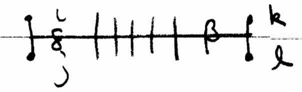
$$= \sum_{i} X^{\gamma}(i,j)\, Y^{\delta}(i,k)\, R(i,j;k,l)$$

X, Y : intertwiner "given"

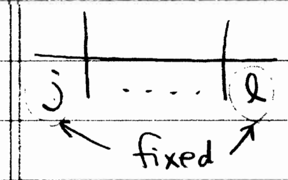
Instead of 8V, we consider SOS model.



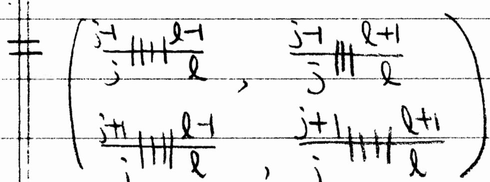
$$\equiv T^{\{\gamma\}}_{\{\alpha\}}(i,j;k,l) = $$



$$= \overset{-1}{Y}_{\delta}(i,j)\, T^{\{\gamma\}\delta}_{\{\alpha\}\beta}\, Y^{\beta}(k,l)$$



$$\equiv T_{jl} = \begin{pmatrix} A_{jl} & B_{jl} \\ C_{jl} & D_{jl} \end{pmatrix}$$



$$= \begin{pmatrix} T(j-1,j;l-1,l) & T(j-1,j;l+1,l) \\ T(j+1,j;l-1,l) & T(j+1,j;l+1,l) \end{pmatrix}$$

<u>state</u>

$$\begin{matrix} l_0 \,|\, l_1 \,|\, l_2 \,|\, \cdots \,|\, \cdot\cdot \,|\, l_{N-1} \,|\, l_N \\ \alpha_1 \quad \alpha_2 \quad \cdots\cdots \quad \alpha_N \end{matrix} \equiv \phi^{\{\alpha\}}_{l_0 l_N} = X^{\alpha_1}_{(l_0,l_1)} \cdots X^{\alpha_N}_{(l_{N-1},l_N)}$$

pseudovacuum ; $l_{i+1} = l_i - 1$

$$l_0 \,|\, l_0-1 \,|\, \cdots \,|\, l_0-N$$

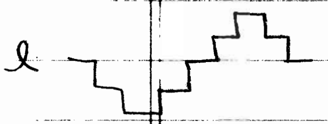
pseudo vacuum. for some $\overset{\text{integer}}{l}$

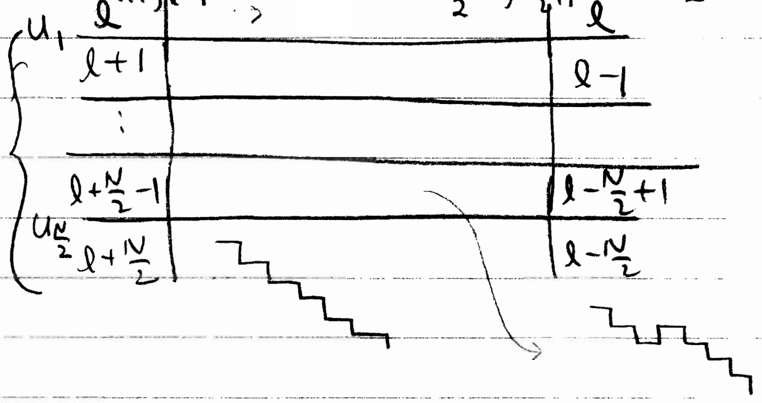
$$\phi_{l+\frac{N}{2},\, l-\frac{N}{2}} = {}_{l+\frac{N}{2}}| \cdots\cdots |_{l-\frac{N}{2}}$$

$$C_{l+\frac{N}{2},\, l-\frac{N}{2}}\ \phi_{l+\frac{N}{2},\, l-\frac{N}{2}} = \begin{array}{l|c|l} l+\frac{N}{2}+1 & \text{impossible} & l-\frac{N}{2}-1 \\ \hline l+\frac{N}{2} & - - - & l-\frac{N}{2} \end{array} = 0$$

act "creation" operator

$$\Psi_l(u_1, \dots, u_{\frac{N}{2}}) = B_{l+1,\, l-1}(u_1) \cdots B_{l+\frac{N}{2}-1,\, l-\frac{N}{2}+1}(u_{N/2-1})\, B_{l+\frac{N}{2},\, l-\frac{N}{2}}(u_{N/2})\, \phi_{l+\frac{N}{2},\, l-\frac{N}{2}}$$





- Transfer matrix of 8 vertex model

take $T^{\{\gamma\}}_{\{\alpha\}}(kl;k,l) = Y^{-1}_{\delta}(k,l)\, T^{\{\gamma\}\delta}_{\{\alpha\}\beta}\, Y^{\beta}(kl)$

tr $\quad$ " $\quad = \mathrm{tr}[Y^{-1}(kl)\, T^{\{\gamma\}}_{\{\alpha\}}\, Y(kl)]$

$\quad\|\quad = \mathrm{tr}\, T^{\{\gamma\}}_{\{\alpha\}} = T_{8V}.$

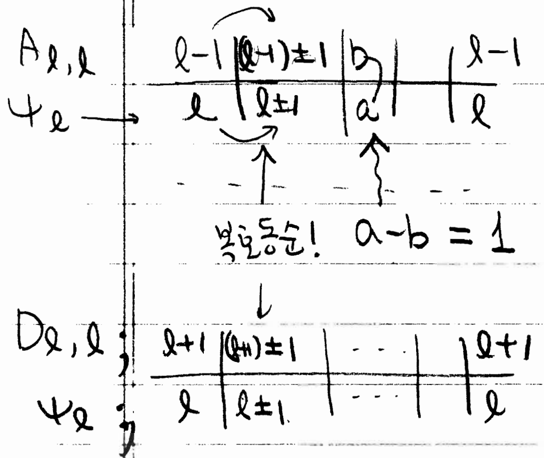
$A_{l,l} = T(l-1, l; l-1, l)$

$+$

$D_{l,l} = T(l+1, l; l+1, l)$ $\qquad$ for any l

(independent of l)

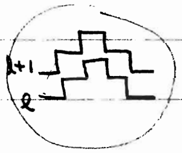
act $(A_{\ell,\ell} + D_{\ell,\ell})$ on Ψ_ℓ



$$= \Psi_{\ell-1} \cdot \lambda_A$$

ℓ-independent

$$= \Psi_{\ell+1} \cdot \lambda_B$$



$$T_{8V} \Psi_\ell = \lambda_A \Psi_{\ell-1} + \lambda_B \Psi_{\ell+1}$$

$T_{8V} \equiv$ SOS

define $\Psi_\theta(u_1, \cdots u_{N/2}) \equiv \sum_{\ell=-\infty}^{\infty} e^{-i\theta\ell} \Psi_\ell(u_1 \cdots u_{N/2})$

$$T_{8V} \Psi_\theta = \sum_\ell e^{-i\theta\ell} \left(\lambda_A \Psi_{\ell-1} + \lambda_D \Psi_{\ell+1}\right)$$

$$= \left[e^{-i\theta} \lambda_A + e^{i\theta} \lambda_D\right] \Psi_\theta$$

& vanishing unwanted terms

$$\boxed{\left(\frac{b}{a}(u_j)\right)^N \prod_{i=1}^{N/2} \frac{a}{b}(u_j - u_i) \frac{b}{a}(u_i - u_j)\, e^{2i\theta} = -1}$$

critical (trigonometric) limit $p \to 0$, $K \to \frac{\pi}{2}$

$$n = \frac{N}{2} , \quad \Theta = m \underbrace{\left(\frac{\pi}{\nu}\right)}_{\text{integer}} = m\gamma , \quad \gamma = \frac{2K}{\nu} \to \frac{\pi}{\nu}$$

$e^{-i\Theta l}$ periodic with ν $\therefore$ $l = 0, \cdots, \nu-1$

$\therefore$ $n = \frac{N}{2}$ mod ν etc

$\Rightarrow$ RSOS