

①

Def. tangle: compact 1-mfd properly embedded in $\mathbb{R} \times \mathbb{R} \times [0, 1]$
 s.t. boundary of it is a set of distinct points
 in $\{0\} \times (\mathbb{R} \times [0, 1])$.

1. nonoriented tangle

Def. ① elementary tangle diagrams



② tensor product

$$T_1 \otimes T_2 = \begin{array}{|c|c|} \hline T_1 & T_2 \\ \hline \end{array}$$

③ composition

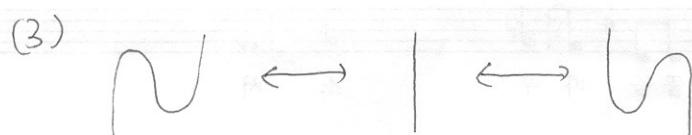
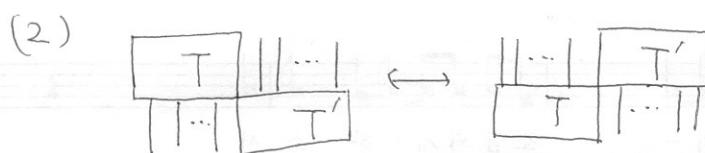
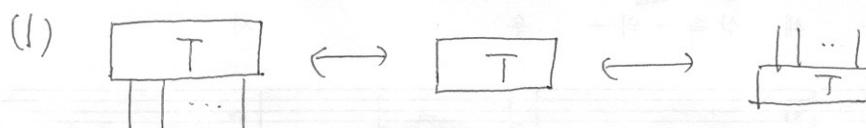
$$T_1 \cdot T_2 = \begin{array}{|c|} \hline T_1 \\ \hline T_2 \\ \hline \end{array}$$

Def. A tangle diagram sliced by horizontal lines

s.t. each domain between adjacent horizontal lines has either a single crossing or a single critical point is called a sliced tangle diagram.

Note) A sliced tangle diagram can be represented by tensor products and compositions of elementary tangle diagrams.

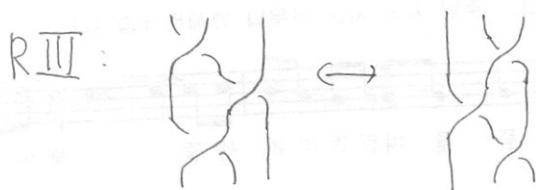
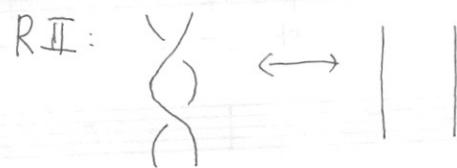
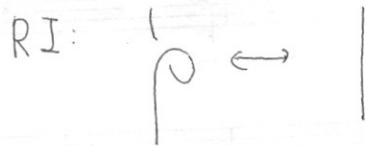
Def.) the Turaev moves (unoriented version)



(2)



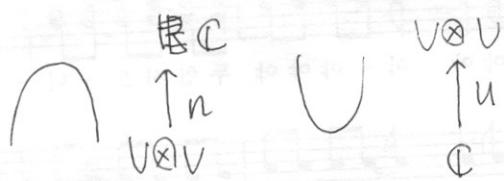
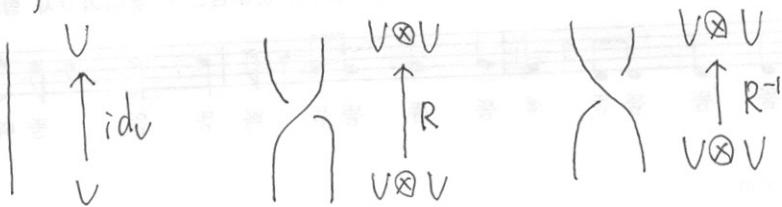
(5) the Reidemeister moves



Thm. T_1, T_2 : tangles, D_1, D_2 : sliced tangle diagrams of T_1, T_2 respectively

T_1 and T_2 are isotopic $\Leftrightarrow D_1$ can be transformed to D_2 by finite number of Turaev moves

Def. The linear maps associated to the elementary tangle diagrams



Note) Let a basis of V be $\{e_i : i=1,2,\dots,n\}$.

Let $n(e_i \otimes e_j) = n_{ij}$, $U(I) = \sum_{i,j} u^{ij} e_i \otimes e_j$. Then from the Turaev moves (3), we obtain

(3)

$$(n \otimes \text{id}_V) \cdot (\text{id}_V \otimes u) = \text{id}_V = (\text{id}_V \otimes n) \cdot (u \otimes \text{id}_V)$$

$$\Rightarrow ① (n \otimes \text{id}_V) \cdot (\text{id}_V \otimes u)(e_k) = (n \otimes \text{id}_V) \cdot \left(\sum_{i,j}^{U^{ij}} e_k \otimes e_i \otimes e_j \right)$$

$$= \sum_{i,j} n_{ki} U^{ij} e_j = e_k \quad \therefore \sum_i n_{ki} U^{ij} = \delta_k^j$$

$$② (\text{id}_V \otimes n) \cdot (u \otimes \text{id}_V)(e_k) = (\text{id}_V \otimes n) \left(\sum_{i,j} U^{ij} e_i \otimes e_j \otimes e_k \right)$$

$$= \sum_{i,j} U^{ij} n_{jk} e_i \quad \therefore \sum_j U^{ij} n_{jk} = \delta_k^i$$

$$\text{So } (n_{ij})^{-1} = (U_{ij} U^{ij})^{-1} \text{ from } ①, ②.$$

This implies that u is determined uniquely by n .

Thm. T : tangle, D : sliced diagram of T

R : invertible endomorphism of $V \otimes V$, n : nondegenerating bilinear form on $V \otimes V$

R and n satisfy

$$① (\text{id}_V \otimes n) \cdot (R^{\pm 1} \otimes \text{id}_V) = (n \otimes \text{id}_V) \cdot (\text{id}_V \otimes R^{\mp 1})$$

$$② n \cdot R = n$$

$$③ (R \otimes \text{id}_V) \cdot (\text{id}_V \otimes R) \cdot (R \otimes \text{id}_V) = (\text{id}_V \otimes R) \cdot (R \otimes \text{id}_V) \cdot (\text{id}_V \otimes R)$$

Let $[D]$ be the linear maps determined by the corresponding from the elementary tangle diagrams to the linear maps associated to them. Then $[D]$ is an isotopy invariant of T .

- pf) The Turaev moves (1), (2), (3), RII, RIII holds trivially by the properties of tensor product and definition of $[D]$.
 ④ and ① guarantee Turaev moves RIII and (4) respectively.
 ② implies

$$\text{Diagram showing two framed tangles, one with a twist and one without, connected by a double-headed arrow labeled (2).}$$

So RI holds by the following diagram:

$$\begin{array}{c} \text{Diagram showing four framed tangles: } \\ \text{1. } \text{Diagram with a twist, labeled (2)} \\ \text{2. } \text{Diagram with a twist and a vertical line, labeled (3)} \\ \text{3. } \text{Diagram with a twist and a vertical line, labeled (4)} \\ \text{4. } \text{Diagram with a twist and a vertical line, labeled (3)} \\ \text{The first three are connected by double-headed arrows labeled (2), (3), and (4) respectively. The fourth diagram is connected to the third by a double-headed arrow labeled (3).} \end{array}$$

Note) We can prove RI ~~is~~ algebraically using the equation below

$$(id_v \otimes n) \cdot (R \otimes id_v) \cdot (id_v \otimes u) \stackrel{(3)}{=} (id \otimes n \otimes n) \cdot (id \otimes id \otimes R \otimes id) \cdot (id \otimes id \otimes id \otimes u \otimes id)$$

$$\stackrel{(4)}{=} (id \otimes n) \cdot (id \otimes id \otimes n \otimes id) \cdot (id \otimes R^{-1} \otimes id \otimes id) \cdot (id \otimes id \otimes id \otimes u) \cdot (u \otimes id)$$

$$\stackrel{(3)}{=} (id \otimes n) \cdot (id \otimes R^{-1}) \cdot \cancel{(u \otimes id)} \stackrel{(2)}{=} (id \otimes n) \cdot (u \otimes id) \stackrel{(3)}{=} id_v$$

Note) ~~If T is a framed tangle, and~~
 지금까지의 내용들은 모두 framed tangle에 대한 것으로 번역이 가능하다. 이를 위해서는 Turaev move의 RI'을 RI'

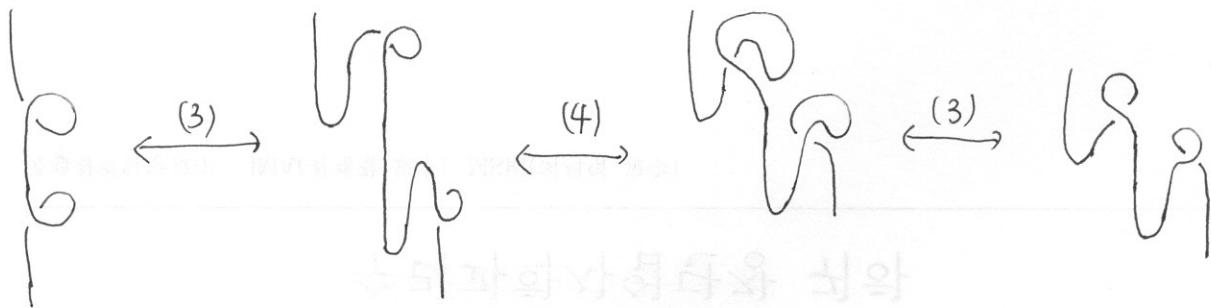
$$\text{RI': } \begin{array}{c} \text{Diagram with a twist, labeled (2)} \\ \leftrightarrow \\ \text{Diagram with a vertical line, labeled (3)} \end{array}$$

로 바꾸고, 위의 thm 을

$$\textcircled{2}' \quad n \cdot R = c \cdot n \quad \text{for some } c \in \mathbb{C} \setminus \{0\}$$

와 같이 바꾸면 된다. 이때 RI'에 대한 증명은 다음과 같이 할 수 있다.

(5)



이때 $n \cdot R = c \cdot n$, $n \cdot R^{-1} = c^{-1} \cdot n$ 이고,

$$\begin{aligned} \text{Diagram } (3) &\longleftrightarrow (\text{id} \otimes n) \cdot (\text{id} \otimes R^{-1}) \cdot (u \otimes \text{id}) = c^{-1} \cdot (\text{id} \otimes n) \cdot (u \otimes \text{id}) \\ &= c^{-1} \cdot \text{id} \end{aligned}$$

$$\begin{aligned} \text{Diagram } (4) &\longleftrightarrow (\text{id} \otimes n) \cdot (\text{id} \otimes R) \cdot (u \otimes \text{id}) = c \cdot (\text{id} \otimes n) \cdot (u \otimes \text{id}) \\ &= c \cdot \text{id} \end{aligned}$$

이로,

$$\begin{aligned} \text{Diagram } (5) &\longleftrightarrow \text{Diagram } (6) \\ \text{Diagram } (5) &\longleftrightarrow \text{Diagram } (4) \end{aligned}$$

이 성립한다.

증명

마지막으로 차원과 대칭성, 주제를 살펴보자.

차원
증명
주제

주제를 살펴보자. 증명과 대칭성(대수적 성질)