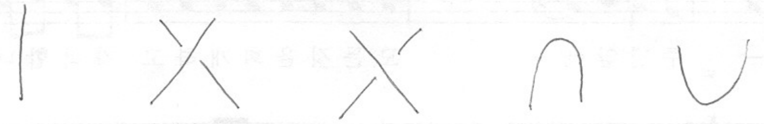
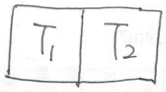
Def. tangle: compact 1-mfd properly embedded in $\mathbb{R}\times\mathbb{R}\times[0,1]$ s.t. boundary of it is a set of distinct points in $\{0\}\times\mathbb{R}\times\{0,1\}$.

1. nonoriented tangle

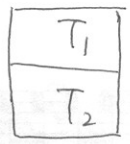
Def. ① elementary tangle diagrams



② tensor product

$T_1 \otimes T_2 =$ 

③ composition

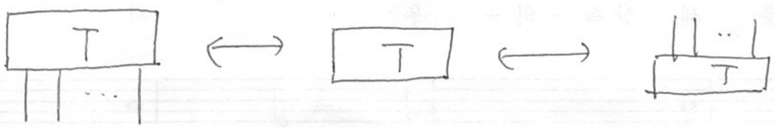
$T_1 \cdot T_2 =$ 

Def. A tangle diagram sliced by horizontal lines s.t. each domain between adjacent horizontal lines has either a single crossing or a single critical point is called a <u>sliced tangle diagram</u>.

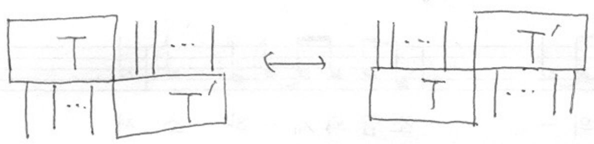
Note) A sliced tangle diagram can be represented by tensor products and compositions of elementary tangle diagrams.

Def) the Turaev moves (unoriented version)

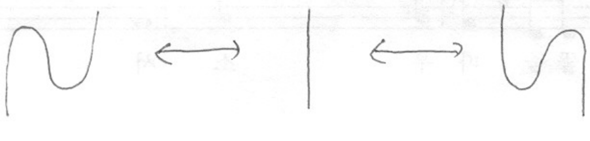
(1)



(2)

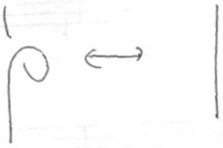


(3)

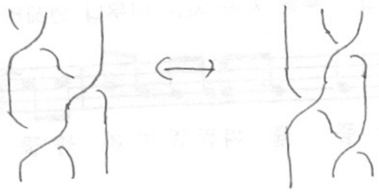


(4) 

(5) the Reidemeister moves

RI: 

RII: 

RIII: 

Thm. T_1, T_2 : tangles , D_1, D_2 : sliced tangle diagrams of T_1, T_2 respectively

T_1 and T_2 are isotopic $\Longleftrightarrow$ D_1 can be transformed to D_2 by finite number of Turaev moves

Def. The linear maps associated to the elementary tangle diagrams

 $V \xrightarrow{id_V} V$

 $V\otimes V \xrightarrow{R} V\otimes V$

 $V\otimes V \xrightarrow{R^{-1}} V\otimes V$

 $V\otimes V \xrightarrow{n} \mathbb{C}$

 $\mathbb{C} \xrightarrow{u} V\otimes V$

Note) Let a basis of V be $\{e_i : i=1,2,\cdots,n\}$.

Let $n(e_i\otimes e_j) = n_{ij}$, $u(1) = \sum_{i,j} u^{ij} e_i\otimes e_j$. Then from the Turaev moves (3), we obtain

$(n\otimes id_V)\cdot(id_V\otimes u) = id_V = (id_V\otimes n)\cdot(u\otimes id_V)$

$\Rightarrow$ ① $(n\otimes id_V)(id_V\otimes u)(e_k) = (n\otimes id_V)\cdot\left(\sum_{i,j} u^{ij}\, e_k\otimes e_i\otimes e_j\right)$

$= \sum_{i,j} n_{ki}u^{ij}e_j = e_k \qquad \therefore \sum_i n_{ki}u^{ij} = \delta^j_k$

② $(id_V\otimes n)(u\otimes id_V)(e_k) = (id_V\otimes n)\left(\sum_{i,j} u^{ij} e_i\otimes e_j\otimes e_k\right)$

$= \sum_{i,j} u^{ij} n_{jk} e_i \qquad \therefore \sum_j u^{ij}n_{jk} = \delta^i_k$

So $(n_{ij})^{-1} = (u^{ij})$ from ①, ②.

This implies that u is determined uniquely by n.

Thm. T: tangle, D: sliced diagram of T

R: invertible endomorphism of $V\otimes V$, n: nondegenerating bilinear form on $V\otimes V$

R and n ~~satisfie~~ satisfy

① $(id_V\otimes n)\cdot(R^{\pm 1}\otimes id_V) = (n\otimes id_V)\cdot(id_V\otimes R^{\mp 1})$

② $n\cdot R = n$

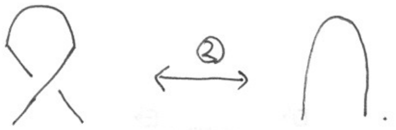
③ $(R\otimes id_V)\cdot(id_V\otimes R)\cdot(R\otimes id_V) = (id_V\otimes R)\cdot(R\otimes id_V)\cdot(id_V\otimes R)$

Let $[D]$ be the linear maps determined by D with the corresponding from the elementary tangle diagrams to the linear maps associated to them. Then $[D]$ is an isotopy invariant of T.

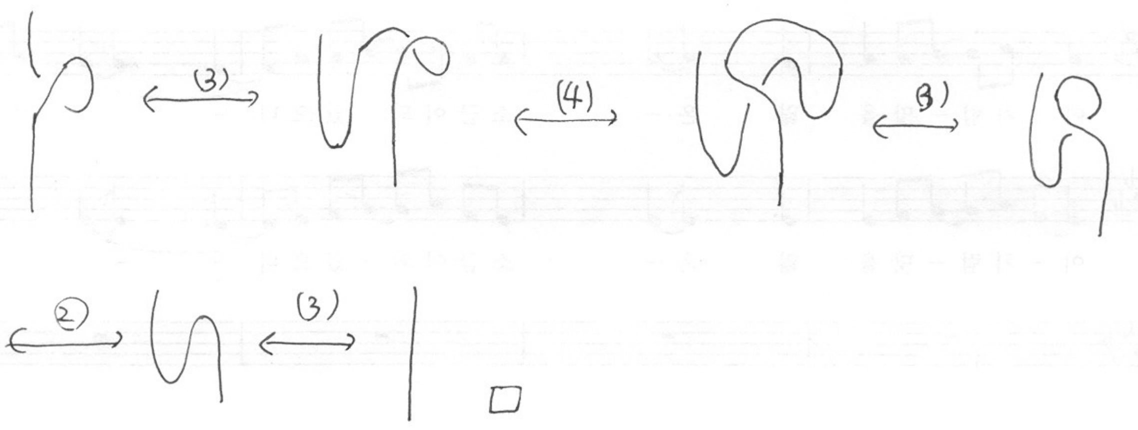
pf) The turaev moves (1), (2), (3), RII, holds trivially by the properties of tensor product and definition of $[D]$.

③ and ① guarantee Turaev moves RIII and (4) respectively.

② implies



So RI holds by the following diagram:



Note) We can prove RI ~~usi~~ algebraically using the equation below

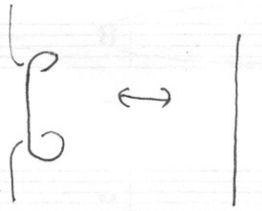
$$(id_V\otimes n)\cdot(R\otimes id_V)\cdot(id_V\otimes u) \overset{(3)}{=} (id\otimes n\otimes n)\cdot(id\otimes id\otimes R\otimes id)\cdot(id\otimes id\otimes id\otimes u)(u\otimes id)$$

$$\overset{(4)}{=} (id\otimes n)(id\otimes id\otimes n\otimes id)\cdot(id\otimes R^{-1}\otimes id\otimes id)(id\otimes id\otimes id\otimes u)(u\otimes id)$$

$$\overset{(3)}{=} (id\otimes n)\cdot(id\otimes R^{-1})(\cancel{id}\; u\otimes id) \overset{②}{=} (id\otimes n)\cdot(u\otimes id) \overset{(3)}{=} id_V$$

Note) ~~If T is a framed tangle, and~~
지금까지의 내용들은 모두 framed tangle에 대한 것으로 변형이 가능하다. 이를 위해서는 Turaev move의 RI을 RI'

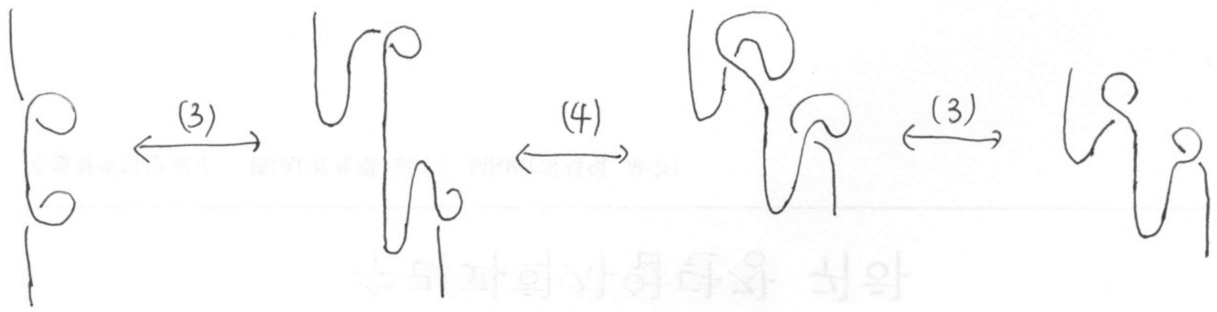
RI':



로 바꾸고, 위의 thm 을

②' $n\cdot R = c\cdot n$ for some $c\in\mathbb{C}\setminus\{0\}$

와 같이 바꾸면 된다. 이때 RI'에 대한 증명은 다음과 같이 할 수 있다.

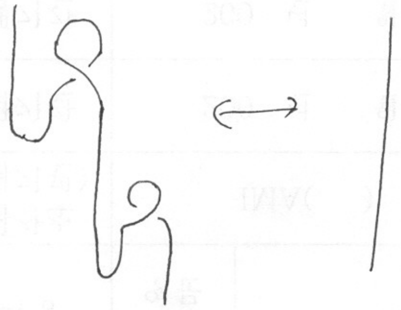


이때 $n\cdot R = c\cdot n$, $n\cdot R^{-1} = c^{-1}\cdot n$ 이고,

$$\longleftrightarrow (id\otimes n)(id\otimes R^{-1})(u\otimes id) = c^{-1}\cdot(id\otimes n)\cdot(u\otimes id) = c^{-1}\cdot id$$

$$\longleftrightarrow (id\otimes n)\cdot(id\otimes R)\cdot(u\otimes id) = c\cdot(id\otimes n)\cdot(u\otimes id) = c\cdot id$$

이므로,



이 성립한다.