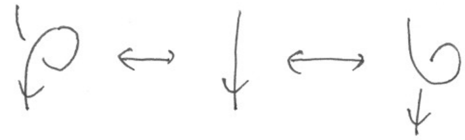
2. Oriented Tangle

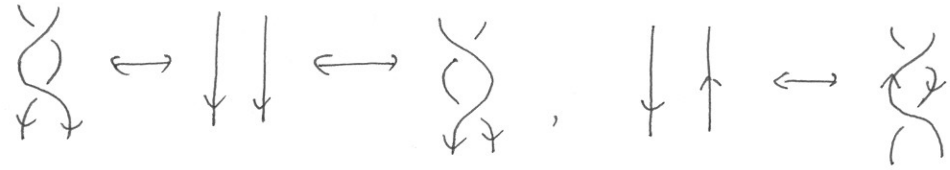
Def. the elementary tangle diagram

Note the Reidemeister moves for oriented diagrams

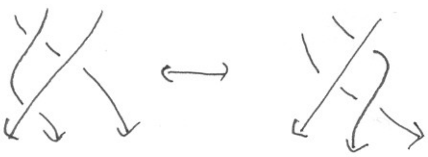
RI :



RII :

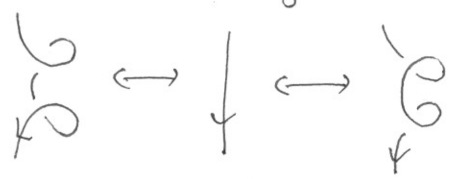


RIII :



(oriented framed link diagram에 대해서는 RI 대신

RI′ :

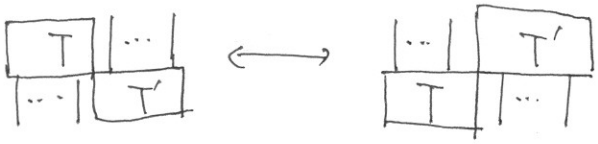


를 생각하면 된다)

Def. the Turaev moves (oriented version)

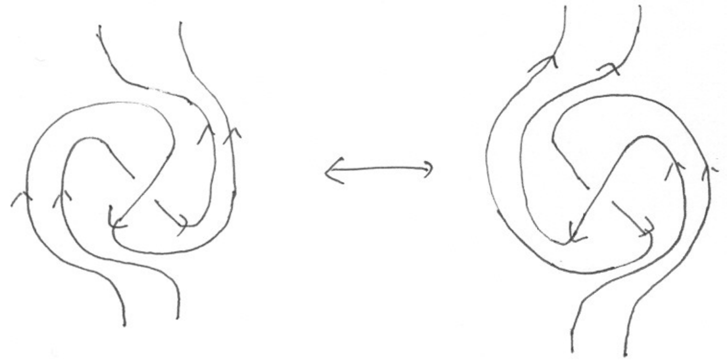
(1)

(2)

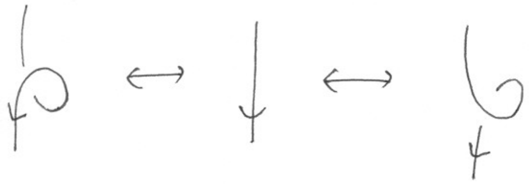


(3)

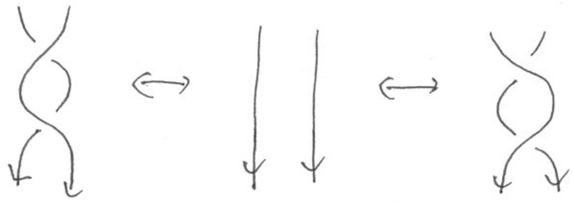
(4)



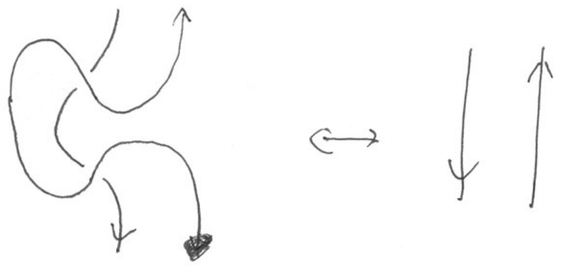
(5)



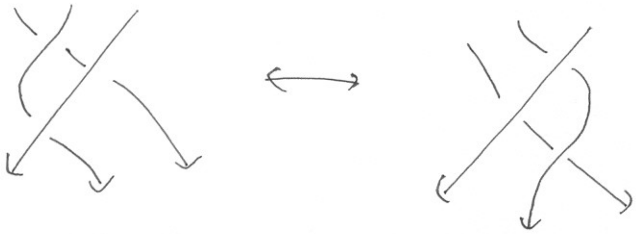
(6)



(7)

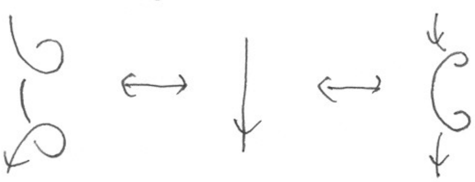


(8)



Note) 위의 (5) ~ (8)은 Reidemeister move에 의하여 필요하며,
(4)는  를 sliced tangle diagram으로 나타내는 서로 다른 두 가지 방법이 존재하기 때문에 필요하다.

Note) framed tangle의 Turaev move는 위의 (5) 대신

(5)':  와 같이 정의한다.

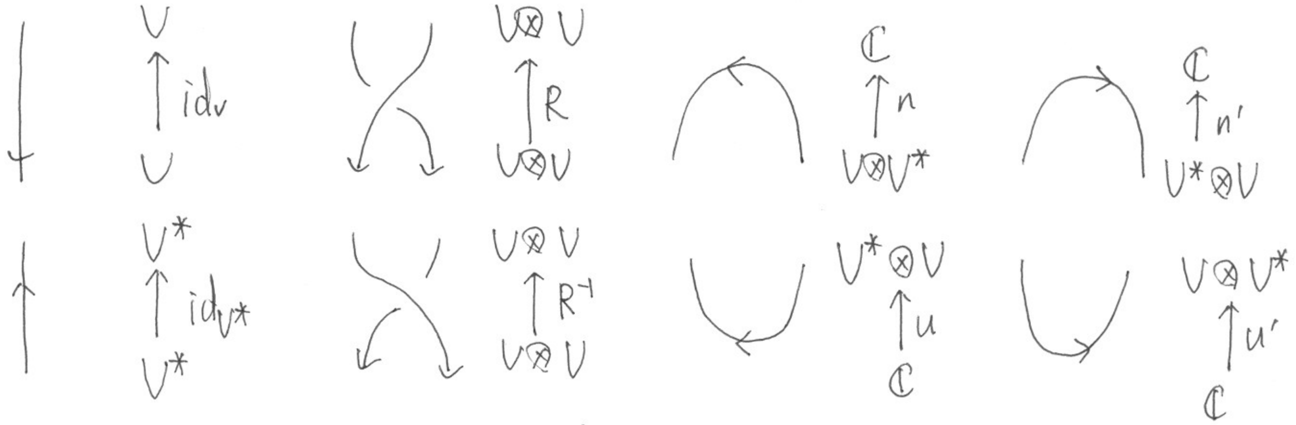
Thm. T_1, T_2 : oriented ~~sliced~~ tangle ~~diagram~~

D_1, D_2 : oriented slied tangle diagram of T_1, T_2 respectively.

T_1 is isotopic to T_2

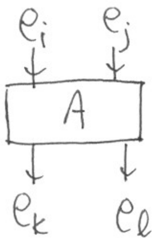
$\Leftrightarrow D_1 \xleftrightarrow[\text{moves}]{\text{Turaev}} D_2$.

Def. The linear maps associated to the oriented elementary tangle diagrams

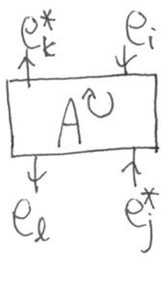


$n(x \otimes f) = f(hx)$, $n'(f \otimes x) = f(x)$, $u(1) = \sum_i e_i^* \otimes (h^{-1} e_i)$, $u'(1) = \sum_i e_i \otimes e_i^*$

Def.

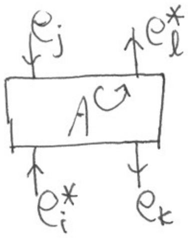


$A(e_k \otimes e_l) = \sum_{i,j} A^{ij}_{kl}\, e_i \otimes e_j$ 라 두자. 이때



$A^{\circlearrowright} : V \otimes V^* \longrightarrow V^* \otimes V$

$A^{\circlearrowright}(e_l \otimes e_j^*) = \sum_{i,k} A^{ij}_{kl}\, e_k^* \otimes e_i$



$A^{\circlearrowleft} : V^* \otimes V \longrightarrow V \otimes V^*$

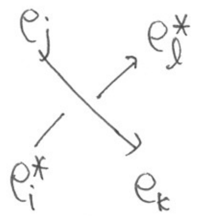
$A^{\circlearrowleft}(e_i^* \otimes e_k) = \sum_{j,l} A^{ij}_{kl}\, e_j \otimes e_l^*$

와 같이 정의한다.

(example) R을 생각하자.

$$R(e_k \otimes e_l) = \sum_{i,j} R^{ij}_{kl}\, e_i \otimes e_j$$

$$R^{\circlearrowright}(e_l \otimes e_j^*) = \sum_{k,i} R^{ij}_{kl}\, e_k^* \otimes e_i$$



$$R^{\circlearrowleft}(e_i^* \otimes e_k) = \sum_{j,l} R^{ij}_{kl}\, e_j \otimes e_l^*$$

Thm. T: oriented tangle, D: sliced diagram of T

$R \in \mathrm{End}(V\otimes V)$, $h \in \mathrm{End}(V)$; invertible maps satisfying

① $R(h\otimes h) = (h\otimes h)R$

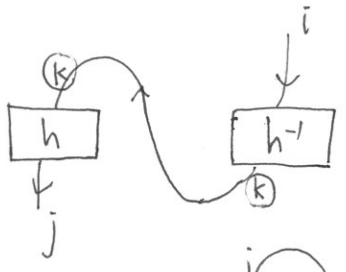
② $Sp_2\left((id_V \otimes h)\cdot R^{\pm 1}\right) = id_V$

③ $(R^{-1})^{\circlearrowleft}\ \left((id_V\otimes h)R(h^{-1}\otimes id_V)\right)^{\circlearrowright} = id_V \otimes id_{V^*}$

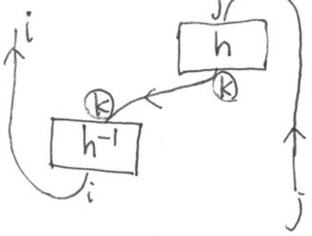
④ $(R\otimes id_V)(id_V\otimes R)(R\otimes id_V) = (id_V\otimes R)\cdot(R\otimes id_V)(id_V\otimes R)$

$\Rightarrow$ $[D]$: isotopic invariant of T

(pf) Turaev move (3)

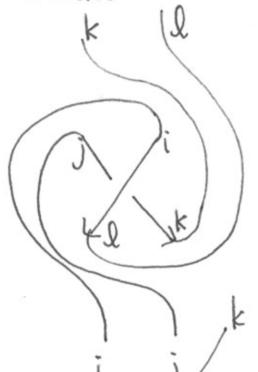


$$\Rightarrow \sum_k h^k_j (h^{-1})^i_k = \delta^i_j$$

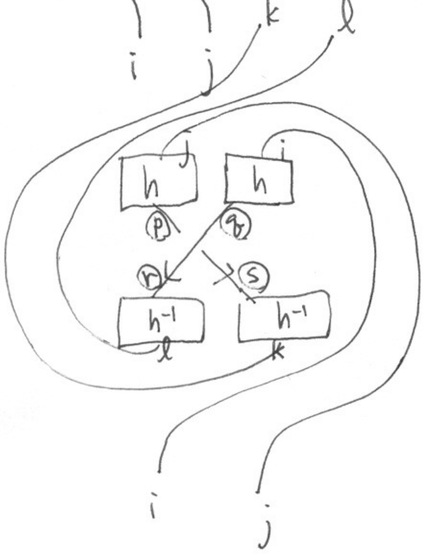


$$\Rightarrow \sum_k (h^{-1})^k_i\, h^j_k = \delta^j_i$$

Turaev move (4)

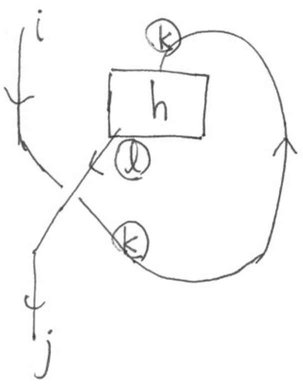


$\Rightarrow R^{ji}_{lk}$



$$\Rightarrow \sum_{p,q,r,s} R^{pq}_{rs} \cdot h^j_p h^i_q (h^{-1})^r_l (h^{-1})^s_k$$
$$\|$$
$$\left((h\otimes h) R (h\otimes h)^{-1}\right)^{ji}_{lk} \overset{①}{=} R^{ji}_{lk}$$

Turaev move (5)



$$\Rightarrow \sum_{k,l} R^{il}_{jk} h^k_l$$

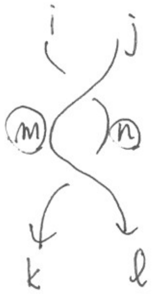
이 때 ②에 의하여

$$e_j = Sp_2\left(R(id\otimes h)\right)(e_j) = \sum_{i,l} \left(R(id\otimes h)\right)^{il}_{jl} e_i$$

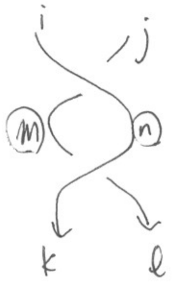
$$\therefore \sum_l \left(R(id\otimes h)\right)^{il}_{jl} = \delta^i_j$$
$$\|$$
$$\sum_{k,l} R^{il}_{jk} h^k_l$$

마찬가지로  $\longleftrightarrow$  로 보일 수 있다.

Turaev moves (6)



$$\Rightarrow \sum_{m,n} R^{ij}_{mn} (R^{-1})^{mn}_{kl} = \delta^i_k \delta^j_l$$

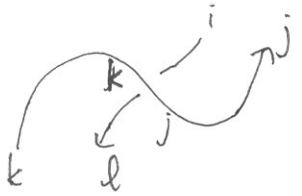


$$\Rightarrow \sum_{m,n} (R^{-1})^{ij}_{mn} R^{mn}_{kl} = \delta^i_k \delta^j_l$$

Turaev moves (7)

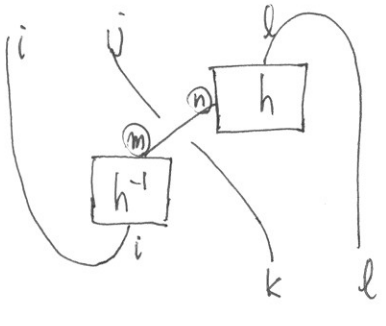
$= C \in \mathrm{Hom}(V^* \otimes V, V \otimes V^*)$

$= D \in \mathrm{Hom}(V \otimes V^*, V^* \otimes V)$ 라 두자.



$$\Rightarrow C^{ij}_{kl} = \cancel{R^{ki}_{lj}} (R^{-1})^{ki}_{lj} = ((R^{-1})^{\curvearrowleft})^{ij}_{kl} \qquad \therefore C = (R^{-1})^{\curvearrowleft}$$

$$\cancel{(R^{-1})^{ki}} \quad (R^{-1})^{ij}_{kl} = ((R^{-1})^{\curvearrowleft})^{jl}_{ik}$$



$$\Rightarrow D^{ij}_{kl} = \sum_{m,n} h^{l}_{n} R^{jn}_{mk} (h^{-1})^{m}_{i}$$

$$= \left((id \otimes h) R (h^{-1} \otimes id)\right)^{jl}_{ik}$$

$$= \left\{\left((id \otimes h) R (h^{-1} \otimes id)\right)^{\circlearrowright}\right\}^{ij}_{kl}$$

$$\therefore D = \left((id \otimes h) R (h^{-1} \otimes id)\right)^{\circlearrowright}$$

이 때 ③에 의해 $C \cdot D = id_V \otimes id_{V^*}$

Turaev moves (8)

이것은 Yang-Baxter equation으로부터 자명하다. $\square$

Note) 만일 T가 framed link였다면, 위의 thm에서 ② 대신

②$'$: $Sp_2((id_V \otimes h)\cdot R^{\pm 1}) = C^{\pm 1}\cdot id_V$

를 쓰면 된다.

<u>example</u>

$$R = \begin{bmatrix} \sqrt{t} & 0 & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & t & \sqrt{t} - t\sqrt{t} & 0 \\ 0 & 0 & 0 & \sqrt{t} \end{bmatrix}, \quad h = \begin{bmatrix} \frac{1}{\sqrt{t}} & 0 \\ 0 & \sqrt{t} \end{bmatrix}$$

는 위의 정리의 ①, ②, ④를 만족시킨다. 그러므로 ③만 성립하면 충분하다.

$$R^{-1} = \begin{bmatrix} \frac{1}{\sqrt{t}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{t}} - \frac{1}{t\sqrt{t}} & \frac{1}{t} & 0 \\ 0 & \frac{1}{t} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{t}} \end{bmatrix}$$ 이므로, $$(R^{-1})^{\circlearrowleft} = \begin{bmatrix} \frac{1}{\sqrt{t}} & 0 & 0 & \frac{1}{\sqrt{t}} - \frac{1}{t\sqrt{t}} \\ 0 & 0 & \frac{1}{t} & 0 \\ 0 & \frac{1}{t} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{t}} \end{bmatrix}$$ 이다.

또한, $$(id_V \otimes h)R(h^{-1} \otimes id_V) = \begin{bmatrix} \sqrt{t} & 0 & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & t & \frac{1}{\sqrt{t}} - \sqrt{t} & 0 \\ 0 & 0 & 0 & \sqrt{t} \end{bmatrix}$$ 이므로

$$((id_V \otimes h)R(h^{-1} \otimes id_V))^{\circlearrowleft} = \begin{bmatrix} \sqrt{t} & 0 & 0 & \frac{1}{\sqrt{t}} - \sqrt{t} \\ 0 & 0 & t & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 0 & \sqrt{t} \end{bmatrix}$$ 가 된다.

그리고 $(R^{-1})^{\circlearrowleft}((id_V \otimes h)\cdot R\cdot(h^{-1} \otimes id_V))^{\circlearrowleft} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ 이 됨은 계산에 의해

확인할 수 있다.

Note) 위의 R, h로부터 얻는 link invariant는 Jones polynomial이다.