Quantum Group Symmetry in QFT

1. Brief introduction to quantum field theory
2. ~~Infinite dimensional symmetric algebra~~
3. Integrable QFT : sine-Gordon model
4. Quantum group symmetry 5. Boundary YBE
6. 3D generalization ; Tetrahedron eq.

---

1. dynamical variable $q_i(t)$ ← time, ← index

"quantum" $[\hat{p}_i(t), \hat{q}_j(t)] = i\hbar\delta_{ij} \neq 0$

$$\underbrace{\hat{q}(t)}_{\text{operator}} \, |q\rangle = \underbrace{q(t)}_{\text{eigenvalue}} \, |q\rangle$$

eigenvalue ↳ measurable

"field"

$$q_i(t) \longrightarrow \Phi(\vec{r}, t)$$

continuous

one dynamical variable per each point in space

"quantum field" $\hat{\Phi}(x_\mu)$, $x_\mu = (\vec{r}, t)$

$$[\hat{\Pi}_{\hat{\Phi}}(\vec{r},t), \hat{\Phi}(\vec{r}',t)] = i\hbar \, \delta^{(D)}(\vec{r} - \vec{r}')$$

$\hat{\Pi}_{\hat{\Phi}}$: generalized momentum

- everything is included in action (Lagrangian)!
- action is defined by symmetry

$$\boxed{\text{Conserved charges}} \longleftrightarrow \boxed{\text{symmetry}} \longleftrightarrow \boxed{\text{phenomenon}}$$

(Nöther thm) S-matrix

$\boxed{\text{algebra}}$ $\boxed{\text{action}}$

Symmetry

$(t, \vec{r})$

$x_\mu \to \Lambda_\mu{}^\nu x_\nu$

* Fundamental ; Lorentz invariance
(Space-time) +
translation invariance $x_\mu \to x_\mu + a_\mu$ $\Big\}$ Poincare P_μ

* Internal symmetry ; gauge symmetry

Coleman-Mandula theorem

Maximal symmetry which is consist with S-matrix
is Poincare + Graded Lie algebra

$\ni$;
$$\{Q^i_\alpha, \bar{Q}^j_\beta\} = \sigma^\mu_{\alpha\beta} P_\mu \delta_{ij}$$

Supersymmetry

if $\underline{\underline{D > 2}}$.

two types of fields:

$$\begin{cases} \phi : \mathbb{R}^D \longrightarrow \mathbb{C} & \text{Bosonic field} \\ \psi : \mathbb{R}^D \longrightarrow \text{Grassmann} & \text{Fermionic field} \end{cases}$$

under Poincare (Lorentz) Group

$$\begin{cases} \phi : \text{singlet} & \text{"scalar"} \\ A_\mu : \text{vector} & \text{"vector"} \\ \psi : \text{spinor} & \end{cases}$$

Dynamics

$$\mathcal{L} = \underset{\text{kinetic}}{T} - \underset{\text{potential}}{V}$$

(ex)

SGM $= \frac{1}{2}(\partial^\mu \phi \, \partial_\mu \phi) + 2\mu \cos\beta\phi$

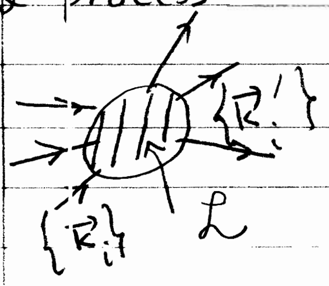
$\boxed{\text{Particle} \leftrightarrow \text{Field duality}}$

Every "particle" is created by quantum field from vacuum

$$\phi(\vec{r}, t) = \int d^d\vec{k} \left(a_{\vec{k}} e^{-i \overbrace{x \cdot k}^{\vec{r}\cdot\vec{k} + \omega \cdot t}} + a^\dagger_{\vec{k}} e^{i x \cdot k} \right), \quad \omega^2 = \vec{k}^2 + m^2$$

$a^\dagger_{\vec{k}} |0\rangle = |\vec{k}\rangle \rightarrow$ a particle with momentum $\vec{k}$

physical process



: S-matrix
(probability amplitude)

2. Infinite dimensional symmetric algebras in physics

Exception of Coleman-Mandula theorem

[D = 2 (1D space + 1D time)]

① Conformal symmetry

Poincare symmetry → stress-energy tensor $T^{\mu\nu}$ is conserved.

$$\frac{\partial}{\partial x^\mu} T^{\mu\nu} = 0 \qquad \begin{cases} T^{00} = \text{Energy} \\ T^{0i} = \text{momentum} \\ T^{ij} = \text{spin} \end{cases}$$

Can add scale invariance.

$$x^\mu \to c \cdot x^\mu \quad \to \underline{J^\mu = x_\nu T^{\mu\nu}}$$

$$\partial_\mu J^\mu = 0 \quad \to \quad T^\mu_{\ \mu} = 0$$

if $\underline{\underline{2D}}$ can introduce $\begin{cases} z = x_0 + i x_1 \\ \bar{z} = x_0 - i x_1 \end{cases}$

$$\begin{cases} \partial_z T^{\bar{z}z} + \partial_{\bar{z}} T^{zz} = 0 \\ T^{\bar{z}z} = 0 \end{cases}$$

$$\partial_{\bar{z}} T^{zz} = 0 \qquad T^{zz} = T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}}$$

form infinite dimensional "Virasoro algebra"

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12}(n^3 - n)\, \delta_{n,-m}$$

"Minimal" model

we impose 'unitarity' of a state of irreducible Rep: $\{|h\rangle, L_{-1}|h\rangle, \cdots\}$

$$\Rightarrow \quad c = 1 - \frac{6}{p(p+1)}, \quad h = \frac{(ps-(p+1)r)^2 - 1}{4p(p+1)}$$

Related to Lattice Model

RSOS

$$\begin{array}{c|c} a & b \\ \hline d & c \end{array} \qquad a, b, c, d = 1, 2, \cdots, p$$

as elliptic modulus $\to 0$; $\longrightarrow$ minimal CFT !

$p = 3$: Ising etc.

2 Yang-Baxter algebra

1) Yangian : rational

2) quantum algebra ✓

3. "Integrable" Quantum field theory (IQFT)

- obtained as "deformed" CFT

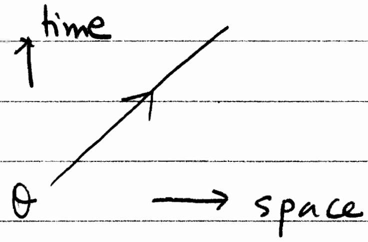
$$\mathcal{L} = \mathcal{L}_{CFT} + \lambda \Phi_{pert}$$

- what is underlying symmetry?

"quantum affine algebra"

1) Origin of YBE

(1+1)D QFT :



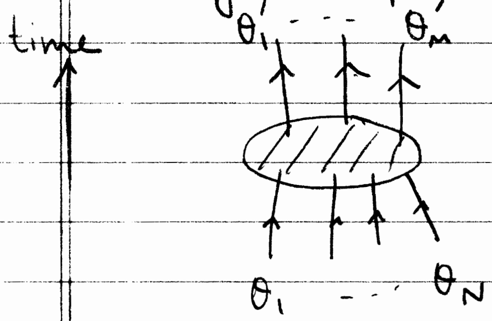
$E = m \operatorname{ch}\theta$

$p = m \operatorname{sh}\theta$

$\boxed{E^2 - p^2 = m^2}$

Einstein formula

Scattering Amplitude ; S-matrix



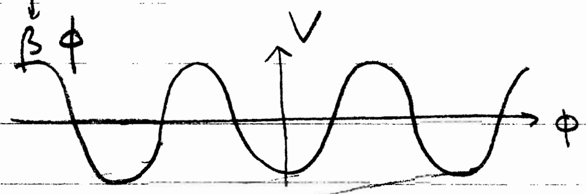
total energy-momentum is preserved but individual no

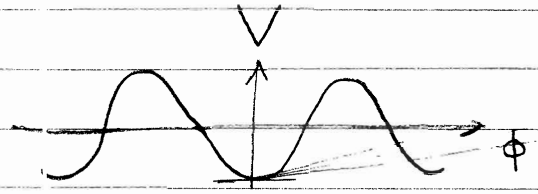
even particle species can change

(ex) Sine-Gordon

coupling constants

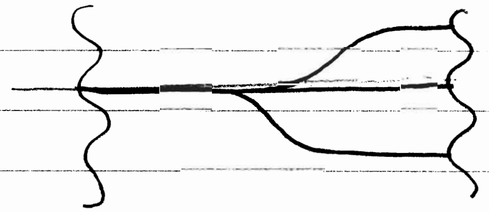
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + 2\mu \cos\beta\phi$$





at $x = \infty$

at $x = -\infty$



has "soliton" solution

ϕ as $x \to \pm\infty$ must be at minima of $-\cos(\beta\phi)$

i.e. $\underline{\beta\phi = 2\pi \cdot n}$ $n = 0, \pm 1 \cdots$

define topological current $J_\mu = \frac{\beta}{2\pi}\varepsilon_{\mu\nu}\partial^\nu\phi$

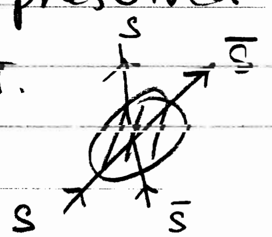
$$\partial^\mu J_\mu = \frac{\beta}{2\pi}\partial^\mu \underbrace{\varepsilon_{\mu\nu}}_{\text{antisym}}\partial^\nu\phi = 0$$

Conserved charge $Q = \int dx\, J_0 = \int dx \frac{\partial}{\partial x}\phi \frac{\beta}{2\pi}$

$$= \frac{\beta\phi}{2\pi}\Big|_{-\infty}^{\infty} = n_\infty - n_{-\infty}$$

if $\begin{cases} Q = +1 \longrightarrow \text{soliton} \quad s \\ Q = -1 \longrightarrow \text{antisoliton} \quad \bar{s} \end{cases}$

Total topological charges are preserved but individual " are NOT.

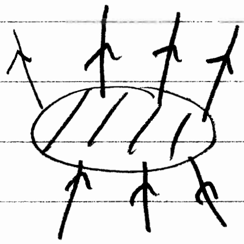


- SGM is a typical IQFT which have infinite # of conserved changes.

$$\partial^\mu J^n_\mu = 0 \qquad n = 0, 1, \cdots, \infty$$

Nöther theorem

$$Q'_n = \int dx\, J^n_0 \quad \text{are preserved}$$



• let's act Q_n on the one particle state $|\theta\rangle$

since Q_n is preserved $\rightarrow$ $Q_n|\theta\rangle = \underbrace{f_n(\theta)}_{\text{eigenvalue}}|\theta\rangle$

[(pf) $\dfrac{dQ_n}{dt} = [H, Q_n] = 0 \rightarrow H, Q_n$ have common eigenstates]

• act on multi particle state $|\theta_1, \theta_2, \cdots, \theta_N\rangle$

initial particles $\quad Q_n|\theta_1\, \theta_2 \cdots \theta_N\rangle = \left[\sum_{i=1}^{N} f_n(\theta_i)\right]|\theta_1 \cdots \theta_N\rangle$

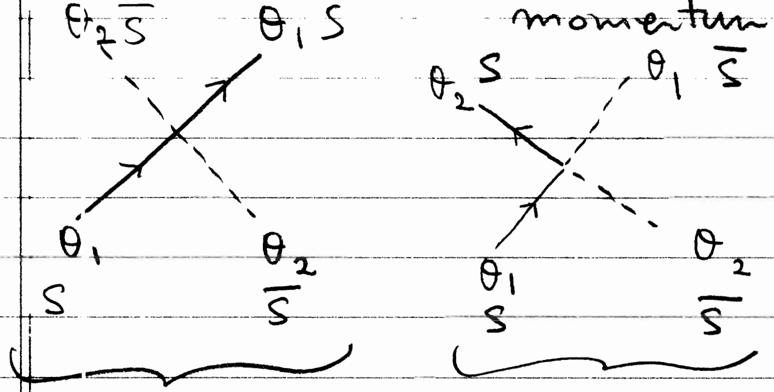
final particles $\quad Q_n|\theta'_1 \cdots \theta'_M\rangle = \left[\sum_{i=1}^{M} f_n(\theta'_i)\right]|\theta'_1 \cdots \theta'_M\rangle$

since Q_n is preserved $\quad \sum_{i=1}^{N} f_n(\theta_i) = \sum_{i=1}^{M} f_n(\theta'_i)$

Notice that there are infinite f_n (usually polynomials of $\mathrm{sh}\,\theta$)

$\therefore$ only solutions are ; $N = M$, $\underline{\{\theta_i\} = \{\theta'_i\}}$

$\{\theta_i\} = \{\theta_i'\}$ means they can exchange momentum.



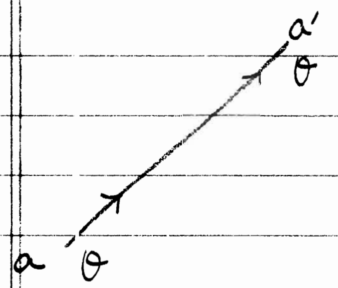
$s(\bar{s})$ preserves momentum

s changes momentum with $\bar{s}$

$\Rightarrow$ nontrivial only when there are several species particles with the same mass ; "mass degeneracy"

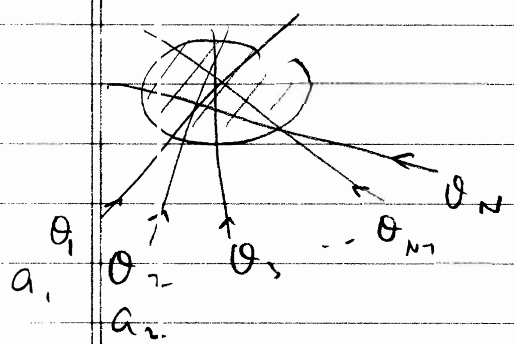
ex:) SGM soliton & antisoliton

- we can say that particle species change while momenta are always same.



$a, a' = s$ or $\bar{s}$

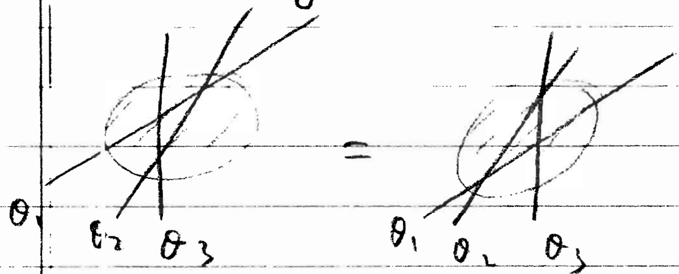
- Since θ's are preserved, there are bunch of straight lines

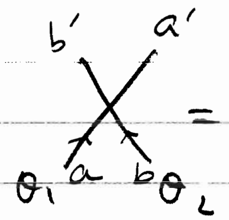


$\{a_1 \cdots a_N\} \rightarrow \{a_1' \cdots a_N'\}$

$a_i = s$ or $\bar{s}$

Consistency Condition ; YBE



for 

$= S_{ab}^{a'b'}(\theta_1-\theta_2)$

- because we know only the initial and final states

- solutions are trigonometric and the same as XXZ because of charge conservation

$$s\,s \to s\,s \qquad \bar{s}\,\bar{s} \to \bar{s}\,\bar{s}$$

4 Quantum Group Symmetry

1. ~~S-matrix level.~~

$$S(\theta) = U(\theta)\begin{pmatrix} sh(\lambda(i\pi-\theta)) & & & \\ & sh\lambda\theta & sh(i\pi\lambda) & \\ & sh\, i\pi\lambda & sh\lambda\theta & \\ & & & sh(\lambda(i\pi-\theta)) \end{pmatrix} \qquad \boxed{\lambda = \frac{8\pi}{\beta^2}-1}$$

can be derived only from the YBE.

- $$S(\theta) = U(\theta)\, P\left[e^{-\lambda\theta}\hat{R}^{+} - e^{+\lambda\theta}\hat{R}^{-}\right]$$

with $\underline{\hat{R}^{\pm} \text{ solution } \overbrace{\text{of YBE}}^{\text{constant}}}$ $\qquad \hat{R}^{+} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$

"universal R-matrix"

$$\hat{R}^{-} \equiv (\hat{R}^{+})^{-1} \qquad \boxed{q = e^{i\pi\lambda}}$$

which satisfy $\quad [\hat{R}^+, \Delta(a)] = 0$

$a \in U_q(sl(2))$

$\Delta : V \to V \otimes V$ "comultiplication"

$$\Delta(h) = 1 \otimes h + h \otimes 1$$

$$\Delta\binom{e}{f} = q^{h/2} \otimes \binom{e}{f} + \binom{e}{f} \otimes q^{-h/2}$$

where $\{e, f, h\}$ form quantum group

$$[f, h] = -2f, \quad [e, h] = 2e, \quad [f, e] = \frac{q^h - q^{-h}}{q - q^{-1}}$$

$$\therefore \quad [S(\theta), \Delta(a)] = 0$$

For rapidity dependence, $\quad e^{\lambda\theta} = x \quad$ we define

$$\hat{e}_0 \equiv x f \qquad \hat{f}_0 = x^{-1} e \qquad \hat{h}_0 = -h$$

$$\hat{e}_1 = x e \qquad \hat{f}_1 = x^{-1} f \qquad \hat{h}_1 = h$$

"Chevalley basis" $\quad$ affine $\hat{sl}(2)$ algebra

$$x^{-1} \hat{R}^+ - x \hat{R}^- \equiv \hat{R}(x)$$

$$[\hat{R}(x), \hat{\Delta}(a)] = 0 \quad , \quad a \in U_q[\hat{sl}(2)]$$

2. Lagrangian Level

Non local conserved current

$$J_{\pm}(x,t) = \exp\left(\pm \frac{4\pi i}{\beta^2} \phi(x,t) \pm \frac{4\pi i}{\beta^2} \int_{-\infty}^{x} dy \; \partial_t \phi(y,t) \right)$$

$$H_{\pm}(x,t) = \exp\left(\pm \left(\frac{4\pi}{\beta^2} - 1 \right) \phi(x,t) \pm \frac{4\pi}{\beta^2} \int_{-\infty}^{x} dy \, \partial_t \phi \right)$$

satisfy $\qquad \partial_{\bar z} J_{\pm} = \partial_z H_{\pm}$

Conserved charge $\qquad Q_{\pm} = \int dz \, J_{\pm} + \int d\bar z \, H_{\pm}$

$$\bar Q_{\pm} = \int d\bar z \, \bar J_{\pm} + \int dz \, \bar H_{\pm}$$

satisfy

$$Q_{\pm} \bar Q_{\pm} - q^2 \bar Q_{\pm} Q_{\pm} = 0$$

$$Q_{+} \bar Q_{-} - q^{-2} \bar Q_{-} Q_{+} = 1 - q^{2J}$$

$$[J, Q_{\pm}] = \pm 2 Q_{\pm}$$

$\Rightarrow$ one can identify

$$Q_{+} = \hat e_1 \, q^{h_1/2} \qquad \bar Q_{+} = \hat f_1 \, q^{h_1/2}$$

$$Q_{-} = \hat e_0 \, q^{h_0/2} \qquad \bar Q_{-} = \hat f_0 \, q^{h_0/2}$$

/

Application of q-group

$\binom{s}{\bar{s}}$ form spin-$\frac{1}{2}$ REP. $V_{\frac{1}{2}}$

Lusztig-Russo Theorem

if $q \neq$ a root of unity, $\mathrm{REP}[\mathcal{U}_q g] = \mathrm{REP}[\mathcal{U} g]$

" if $q^N \neq 1$, REP of $sl_q(2) \equiv sl(2)$

We can consider multi-soliton states

$$V_{\frac{1}{2}} \otimes V_{\frac{1}{2}} \otimes \cdots \otimes V_{\frac{1}{2}}$$

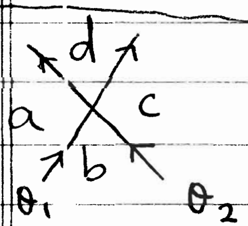
Using Clebsch-Gordan Coeff, we can decompose

$$= \bigoplus_j V_j$$

This is the same as 6 Vertex $\longleftrightarrow$ SOS model

Instead of $s, \bar{s}$, we have kinks $K_{ab}(\theta)$ $\quad a \,|_{\frac{1}{2}}\, b$

$$K_{ab}(\theta_1) K_{bc}(\theta_2) = S\begin{pmatrix} a & b \\ d & c \end{pmatrix}(\theta)\, K_{ad}(\theta_2) K_{dc}(\theta_1)$$



Special thing happens when q is a root of unity

q-CGs are expressed by q-numbers

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

q-CG $\ni [2a+1]$, if $q = e^{\frac{i\pi}{p}}$ $\left(\lambda \equiv \frac{1}{p} \right)$

$$1 \leq 2a+1 < p$$

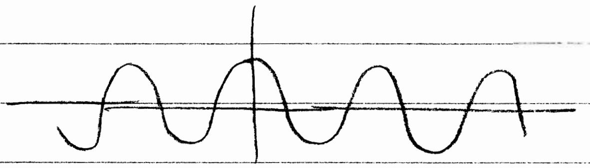
$$\therefore \boxed{0 \leq a \leq \frac{p}{2} - 1}$$

or $a_{max} = \frac{p}{2} - 1$

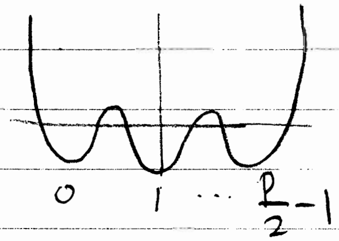
i.e $\boxed{\bigoplus_{j=0}^{\frac{p}{2}-1} V_j}$ $\longrightarrow$ truncate $j > \frac{p}{2} - 1$

analogy of

RSOS $\longrightarrow$ Restricted SG model



$\longrightarrow$

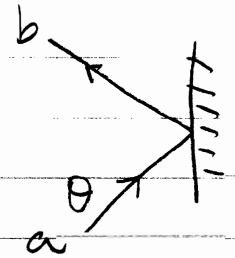


scale invariant

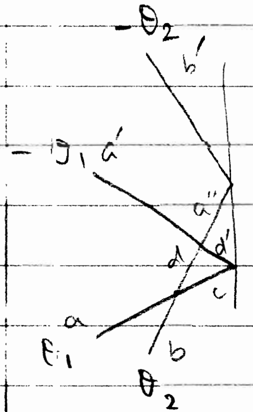
$M_p + \lambda \Phi_{pert}$

CFT

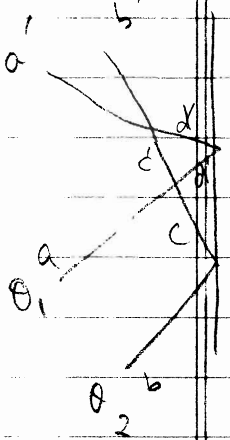
5 BYBE

introduce Boundary 

$\equiv R_{ab}(\theta)$



$$\sum_{cdd'a''} S_{ab}^{cd}(\theta_1-\theta_2)\, R_{c}^{d'}(\theta_1)\, S_{dd'}^{a'a''}(\theta_2+\theta_1)\, R_{a''}^{b'}(\theta_2)$$



$$= \sum R_{b}^{c}(\theta_2)\, S_{ac}^{d'c'}(\theta_1+\theta_2)\, R_{d}^{d'}(\theta_1)\, S_{c'd'}^{b'a'}(\theta_1-\theta_2)$$

- GVertex / XXZ / SGM $\longrightarrow$ Ghoshal & Zamol
- SOS / RSOS $\longrightarrow$ Ahn, Koo

$$R(\theta) = \begin{pmatrix} \cosh(\xi+\theta) & k\, \mathrm{sh}\, 2\theta \\ k\, \mathrm{sh}\, 2\theta & \mathrm{ch}(\xi-\theta) \end{pmatrix}$$

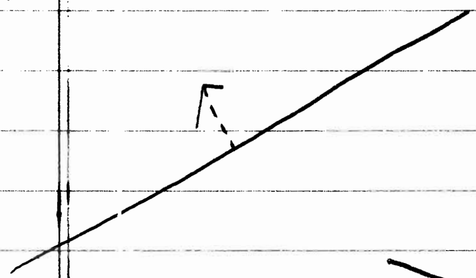
No mathematic structure found yet

But boundary Yangian is studied

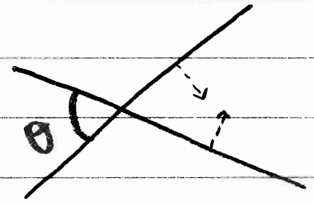
5. 3D Integrable lattice / (2+1)D QFT

① Zamododchikov model

(rigid, infinite) string moving on 2D plane

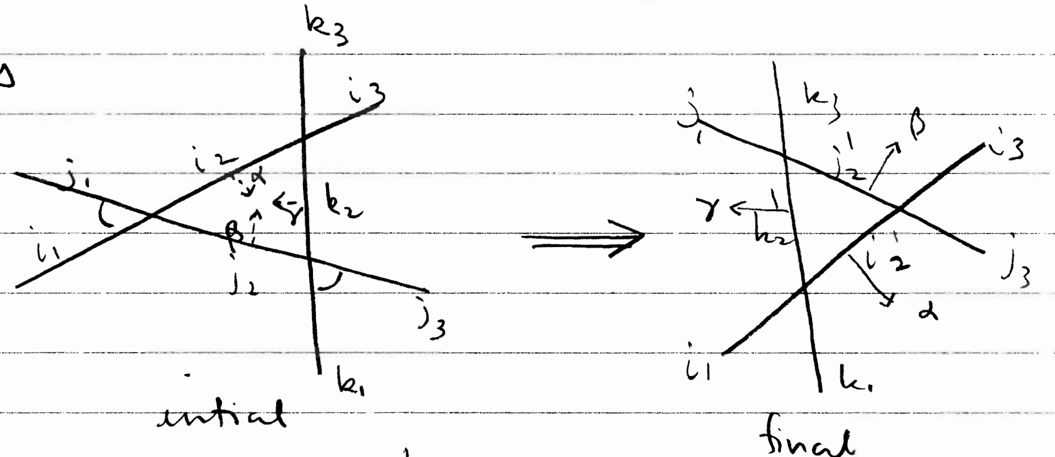


two strings



Since the lengths are infinite, this configuration is invariant.

three strings

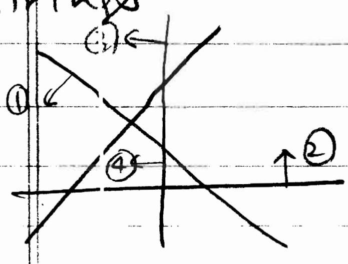


"vertex model"

$$S^{k_1\, k_2\, k_3\, k_2'}_{\substack{j_1\, j_2\, j_3\, j_2' \\ i_1\, i_2\, i_3\, i_2'}}(\alpha, \beta, \gamma)$$

should satisfy "tetrahedron Eq"

four strings



= another process (see figure) Hietarinta

SSSS = SSSS

2-color model $\quad i = \pm 1$

Zamolodchikov found a solution of the

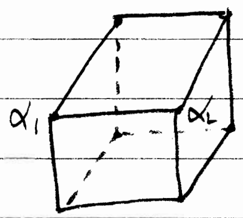
$$\frac{2^{12}}{2 \cdot 2} = 1024 \text{ unknowns !}$$

$\pm$ time reversal

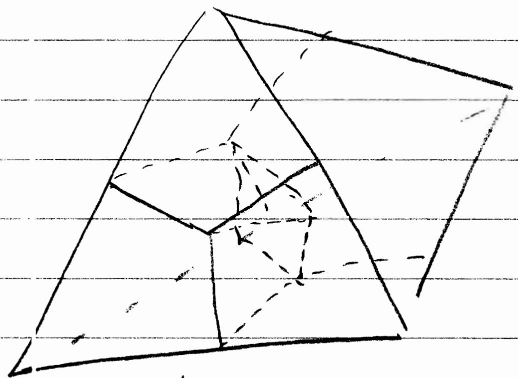
See (17)

② Baxter version of Zamolodchikov model.

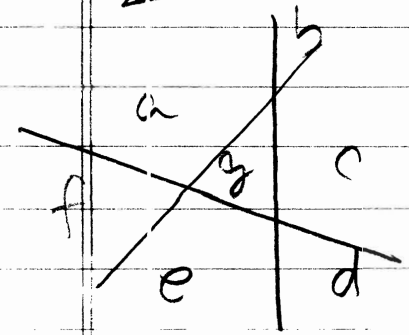
3D lattice model



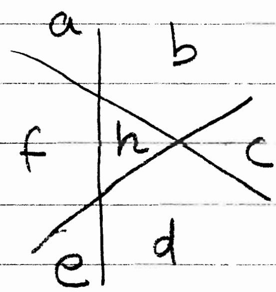
$$= W(\underbrace{\alpha_1 \cdots \alpha_8}_{\pm})$$



See figure (Baxter)



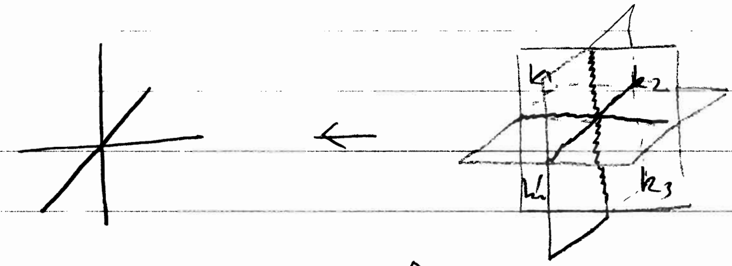
$\longrightarrow$



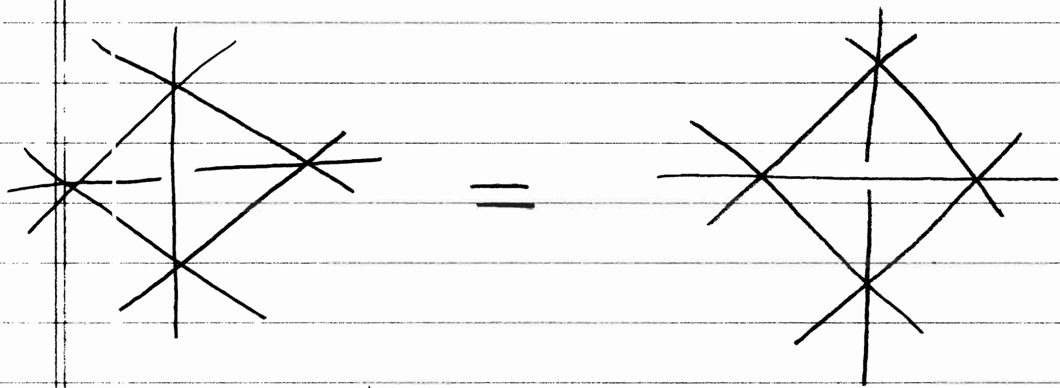
$\rightarrow 2^8$

→ (18) //

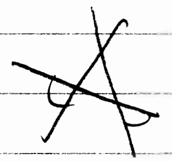
Zamolodchikov ~~model~~ in 3D lattice

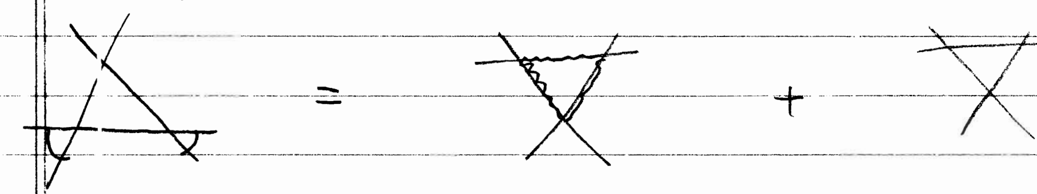


12 faces



tetrahedron eq.

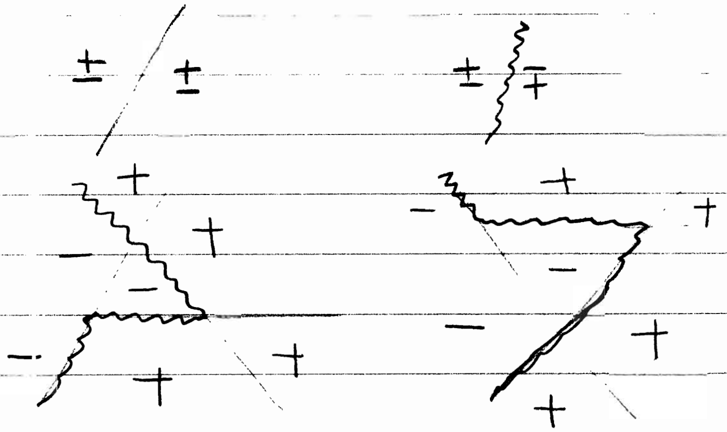
impose rapidity as two angles of 
and independent of speeds.



etc and found the solutions.

back to 16

Relation to Zamolodchikov.



etc.

∴ two models are equivalent