Kirby Calculus

Basic References

1. Ohtsuki : Quantum invariants chap 8.
2. Gomp & Stipsicz : 4-manifolds and Kirby calculus chap 4 & chap 5.
3. Fenn & Rourke : On Kirby's calculus of links
4. Prasolov & Sossinsky : Knots, links, braids and 3-manifolds.

0. Introduction

- 3-manifold 의 topological invariant

$$I : \{3\text{-manifold}\} \longrightarrow S : \text{some well known set}$$

such that " $M \underset{\phi}{\cong} M' \Rightarrow I(M) = I(M')$ ".

homeomorphism.

- <u>Thm</u> (Lickorish) Any closed connected orientable 3-manifold can be obtained from S^3 by integral surgery along some framed link.

- <u>Thm</u> (Kirby) For framed links L and L' in S^3

$$S^3_L \cong S^3_{L'} \quad \text{iff} \quad L \underset{\phi}{\sim} L'$$

$$\phi : \begin{cases} \text{isotopy} \\ K\text{I}, K\text{II} \ (\text{so called Kirby moves}) \end{cases}$$

<u>Thm</u> (Fenn-Rourke) For framed links L and L' in S^3,

$$S^3_L \cong S^3_{L'} \quad \text{iff} \quad L \underset{\phi}{\sim} L'$$

$$\phi : \begin{cases} \text{isotopy} \\ \text{FR move.} \ (\text{Fenn-Rourke move}) \end{cases}$$

(blow-ups / downs)

<u>Remark</u> : 1. framed link 는 위의 두 경우에 integral framing을 갖는 것으로 생각함.

2. 만일 rational framing을 갖는 경우 우리는 다음과 같은 statement를 얻음.

<u>Thm</u> L, L' : links with rational coefficients in S^3

$$S^3_L \cong S^3_{L'} \quad \text{iff} \quad L \underset{\phi}{\sim} L'$$

$$\phi : \begin{cases} \cdot \ \text{isotopy} \\ \cdot \ \text{Rolfsen twist} \\ O^{\infty} \leftrightarrow \emptyset . \end{cases}$$

1. 3-manifolds and their surgery presentations

Def'n. The image of $\coprod_{i=1}^{n} S^1 \hookrightarrow S^3$ is called knot / link.

Def'n. $K_1, K_2 \subset S^3$ oriented knot.

$$H_2(S^3, S^3 \setminus K_1 : \mathbb{Z}) \overset{\text{excision}}{\cong} H_2(\nu(K_1), \partial\nu(K_1)) \cong \mathbb{Z}.$$

$$H_1(S^3 \setminus K : \mathbb{Z}) \cong \langle \mu_K \rangle \cong \mathbb{Z}. \quad \text{where } \mu_K = \text{meridian of } K.$$

① If $[K_2] = n[\mu_K]$, then $lk(K_1, K_2) = n$

or

② $lk(K_1, K_2) = \frac{1}{2} \sum_{\text{crossings}} \mathrm{sign}\left(\times^{K_1}_{K_2} \right)$ where $\begin{cases} \mathrm{sign}(\times) = +1 \\ \mathrm{sign}(\times) = -1. \end{cases}$

Def'n. For a framed knot $(K, v) \subset \partial D^4 = S^3$, the framing coefficient is the $lk(K, K')$ where K' = parallel copy of K determined by v and the orientation of K, K' are chosen to be parallel.

- K' is the framing.

Remark:
- $bb(K) = w(K)$ = signed number of self crossings of K
- The 0-framing is obtained by the outward normal to any oriented Seifert surface

Dehn surgery.

$K \subset S^3$: framed knot. with frame K'

$$S^3_K = (S^3 \setminus \nu K) \cup_{\varphi} (D^2 \times S^1) \quad \text{where } \varphi : \partial(D^2 \times S^1) \to \partial(S^3 \setminus \nu K) : \text{homeo.}$$

is defined by $\varphi(\partial D^2 \times pt) = K'$.

Similarly, $L \subset S^3$: framed link in S^3.

$$S^3_L = (S^3 \setminus \nu L) \cup_{\varphi} \left(\amalg (D^2 \times S^1) \right)$$

In general, M : 3-manifold, $L \subset M^3$ a framed link, i.e the image of an embedding of a disjoint union of annuli into M,

$$M_L = (M \setminus \nu(L)) \underset{\varphi}{\cup} (\amalg (D^2 \times S^1))$$

<u>Remark</u>: M_K is uniquely determined up to homeomorphism by the framed knot.

$$(\because \quad M_K = (M \setminus \nu(K)) \underset{\varphi}{\cup} (D^2 \times S^1), \qquad \varphi(\partial D^2 \times pt) = K'$$

$$= \left((M \setminus \nu(K)) \underset{\varphi}{\cup} (D^2 \times D^1) \right) \cup B$$

and $(M \setminus \nu(K)) \underset{\varphi}{\cup} (D^2 \times D^1)$ is uniquely determined by $\varphi(\partial D^2 \times pt) = K'$.

<u>Lemma</u>. N : 3-manifold with $\partial N \cong S^2$ and B^3 be the 3-ball.

Let $f, g : \partial N \longrightarrow \partial B^3$: two homeo.

then $N \underset{f}{\cup} B^3 \cong N \underset{g}{\cup} B^3$.

(pf) $B^3 = (S^2 \times [0,1]) / \sim$, where $\sim$ collapses $S^2 \times \{1\}$

define $\phi : B^3 \to B^3$ by $\phi(x, t) = (g \circ f^{-1}(x), t)$. for $x \in S^3$, $t \in [0,1]$

Then $\phi|_{\partial B^3} = g \circ f^{-1}$.

$\therefore \ \psi : N \underset{f}{\cup} B^3_3 \longrightarrow N \underset{g}{\cup} B^3$ given by $\psi|_N = id_N$, $\psi|_{B^3} = \phi$

gives a homeomorphism. //)

$\therefore$ We can attach the 3-handle B^3 to $(M \setminus \nu(K)) \underset{\varphi}{\cup} (D^2 \times D^1)$ uniquely up to homeo. //)

<u>Def'n</u> M : a given 3-manifold. $L \subset S^3$ a framed link.

L (together with S^3) is a <u>surgery presentation</u> of M

if $M \cong S^3_L$.

<u>Remark</u>. $H_1(\partial(\nu K);\mathbb{Z}) \cong \langle \mu_K \rangle \oplus \langle \lambda_K \rangle$ where μ_K : preferred meridian of K

λ_K : preferred longitude of K.

$\therefore\ [\varphi|_{\partial D^2 \times pt}] = p\mu_K + q\lambda_K$ for some $p, q \in \mathbb{Z}$.

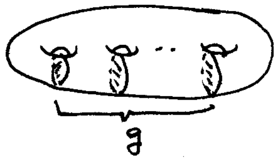
The ratio $p/q \in \mathbb{Q} \cup \{\infty\}$ is called the <u>Dehn surgery coefficient</u> or <u>slope</u>.

<u>Lickorish's Theorem</u>

Any closed connected oriented 3-manifold can be obtained from S^3 by integral surgery along some framed link.

F_g = closed orientable surface of genus g

H_g = 3-manifold w/ $\partial H_g = F_g$



M : 3-manifold such that $M \cong H_g \cup_f H_g$, $f: F_g \longrightarrow F_g$ homeo.

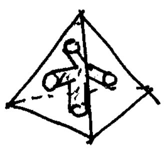
then $H_g \cup_f H_g$ is a Heegaard splitting of M.

<u>Lemma</u>. Any closed connected orientable 3-manifold has a Heegaard splitting.

($\because$ $\exists$ triangulation of M.

Consider $\overline{\nu(\text{1-skeleton})}$ & $\overline{\nu(\text{1-skeleton of the dual triangulation})}$.

It gives a Heegaard splitting.



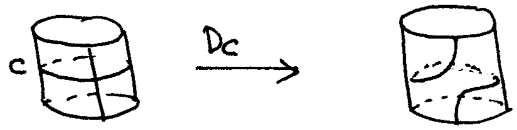
)

Dehn twist

$C \subset F_g$: a simple closed curve.

$\nu(C)$ in F_g is identified with annulus.

A Dehn twist along C is a homeomorphism of F_g to itself which is identity outside the tubular nbd and is equal to the map



(Right handed Dehn twist)

Lemma (Lickorish) $f : F_g \circlearrowleft$ orientation preserving homeo.

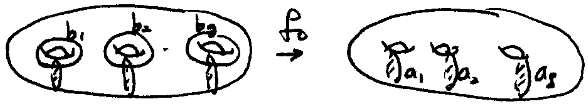
$$\Rightarrow \quad f = D_{C_n}^{\varepsilon_n} \circ D_{C_{n-1}}^{\varepsilon_{n-1}} \circ \cdots \circ D_{C_1}^{\varepsilon_1}$$ for some simple closed curves $C_i \subset F_g$ and $\varepsilon_i = +1$ or $\varepsilon_i = -1$.

Proof of the Lickorish's Thm)

Fix a Heegaard splitting $M \cong H_g \underset{f}{\cup} H_g$, $f : F_g \circlearrowleft$ homeo.

For the same H_g, fix $S^3 = H_g \underset{f_0}{\cup} H_g$

where $f_0(b_i) = a_i$ for $i = 1, \ldots, g$



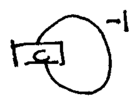
Then $M \cong H_g \underset{f_0^{-1} \circ f}{\cup} (F \times [0,1]) \underset{f_0}{\cup} H_g$

and $f_0^{-1} \circ f : F_g \circlearrowleft$ homeo, so $\exists\, c_1, \ldots, c_n \subset F_g$ s.c.c

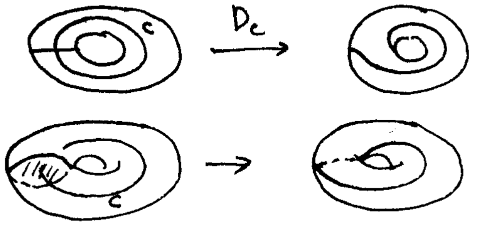
s.t. $f_0^{-1} \circ f = \tau_n \circ \tau_{n-1} \circ \cdots \circ \tau_1$, $\tau_i = D_{C_i}^{\varepsilon_i}$.

$$\therefore \; M \cong H_g \underset{\tau_1}{\cup} (F \times [0,1]) \underset{\tau_2}{\cup} (F \times [0,1]) \cup \cdots \underset{\tau_n}{\cup} (F \times [0,1]) \underset{f_0}{\cup} H_g$$

Claim D_c can be replaced by



($\because$



⑥

$\therefore$ M is obtained from S^3 by doing $-\varepsilon_i$ surgery on each s.c.c. C_i, $i=1, \cdots, n$.

$\therefore$ Any 3-manifld (closed connected oriented) can be obtained from S^3 by integral surgery along some framed link. //)

<u>Proof of the Lemma)</u>

<u>Thm in general form)</u> F: compact oriented 2-mfd w/ boundary ∂F

$\Rightarrow$ any homeo $f: F \to F$ identical on ∂F is isotopic to a composition of Dehn twists.

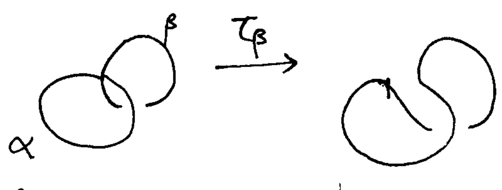
(In the case when $\partial F = \phi$, we need f: orientation preserving)

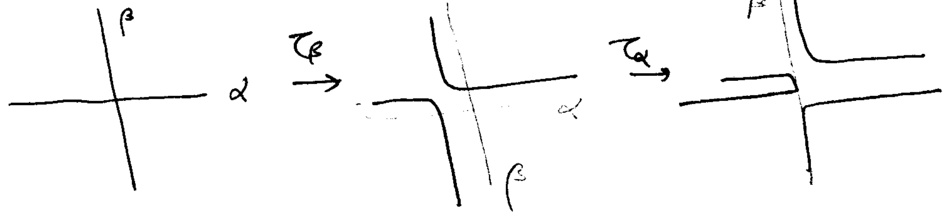
<u>Lemma</u>: $\alpha, \beta \subset F$: closed curve each of which does not disconnect F.

Then $\exists$ a c-homeomorphism (i.e. any composit of Dehntwists &. homeo. isotopic to identity) of F taking α to β.

Pf) <u>CASE 1.</u> $|\alpha \cap \beta| = 1$ pt.

$\tau_\alpha \circ \tau_\beta$:



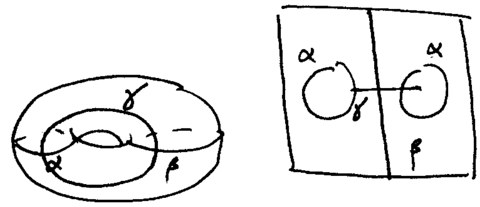


$\therefore \tau_\alpha \circ \tau_\beta(\alpha) \simeq \beta$.

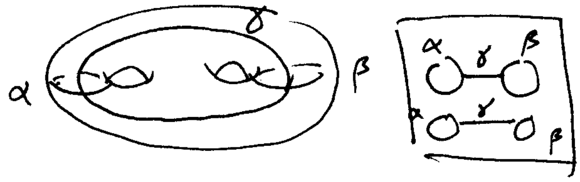
<u>CASE 2</u> $\alpha \cap \beta = \phi \Rightarrow \exists$ oriented curve γ that does not disconnect F and intersects each of the curve α and β transversely at one point.

$\therefore \quad \alpha \xrightarrow{\tau_\alpha \circ \tau_\gamma} \gamma \xrightarrow{\tau_\gamma \circ \tau_\beta} \beta$

① If $\alpha \cup \beta$ disconnects F, then



② If $\alpha \cup \beta$ does not disconnect,



<u>CASE 3</u> $|\alpha \cap \beta| > 1$ (we may assume $|\alpha \cap \beta| \overset{=n}{<} \infty$)

$\exists$ a curve γ such that

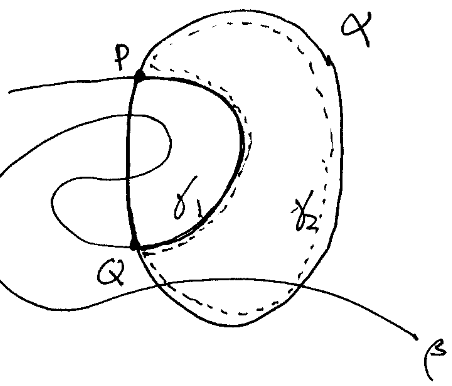
1) γ does not disconnect the surface F
2) γ intersects α at no more than one point
3) γ intersects β at less than n pts.

$(*)$

If so, then we can transform $\alpha \xrightarrow[\phi]{} \gamma$
Case 1 or Case 2.

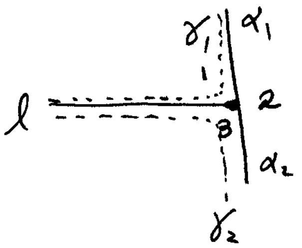
and then induction on n to transform γ into β.

Proof of $(*)$



<u>claim</u> at least one of the curves γ_1 and γ_2 does not disconned the surface F

($\because$ If not,



then it is impossible to get from the region 1 to region 2

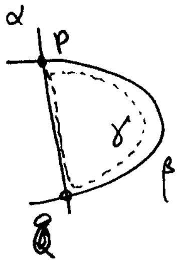
& from region 3 to region 2.

$\therefore$ region 2 is not connected to region 1 or 3.

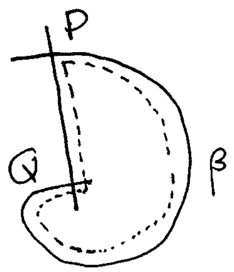
$\therefore$ α disconnects F $\divideontimes$)

Assume γ_1 does not disconnect F.

Push it off itself slightly. $\Rightarrow$ obtain the needed curve γ



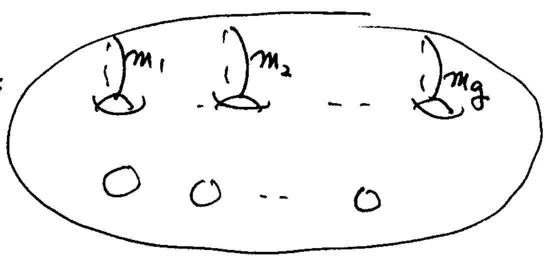
or



//

<u>Proof of the Thm</u>

F $\cong$



Cut F along $m_1, m_2 \cdots m_g$.

$\Rightarrow$ get a disk w/ $k+2g-1$ little disks removed.

Under the homeo $h: F \longrightarrow F$, $m_1 \longmapsto h(m_1)$ that does not disconnect F.

Lemma $\Rightarrow$ $\exists$ c-homeo $f_1 : F \circlearrowleft$ such that $f_1(h(m_1)) = m_1$.

If the orientation of $f_1(h(m_1))$ and m_1 coincide.

$\Rightarrow$ $\exists$ a homeo f_1' isotopic to f_1 for which

$f_1' h$ is the identity on m_1.

After cut along $m_1 - , m_g$, we get a homeo

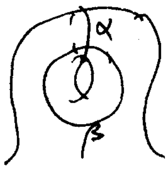
$f_g' \cdots f_1' \circ h$ of the disk with holes, identical on the boundary components.

<u>Recall</u> The group H_n of homeo of the disk with n holes (up to isotopy) is generated by a finite number of twists along closed curves in this disk

$\Rightarrow$ $f_g' - - f_1' \circ h$ is isotopic to a composition of Dehn twists. and so is h. //

If $f_1 h(m_1)$ and m_1 have opposite orientations.

Since $\alpha = m_1$ does not disconnect F, $\exists$ a curve β intersecting α at exactly at one pt.



Let α^{-1} be the α with opposite orientation

Then $\tau_\beta \tau_\alpha \tau_\beta$: $\alpha \longmapsto \beta$, $\beta \longmapsto \alpha^{-1}$ respectively.

$\therefore$ $(\tau_\beta \tau_\alpha \tau_\beta)^2$, $\alpha \longrightarrow \alpha^{-1}$, $\beta \longrightarrow \beta^{-1}$.

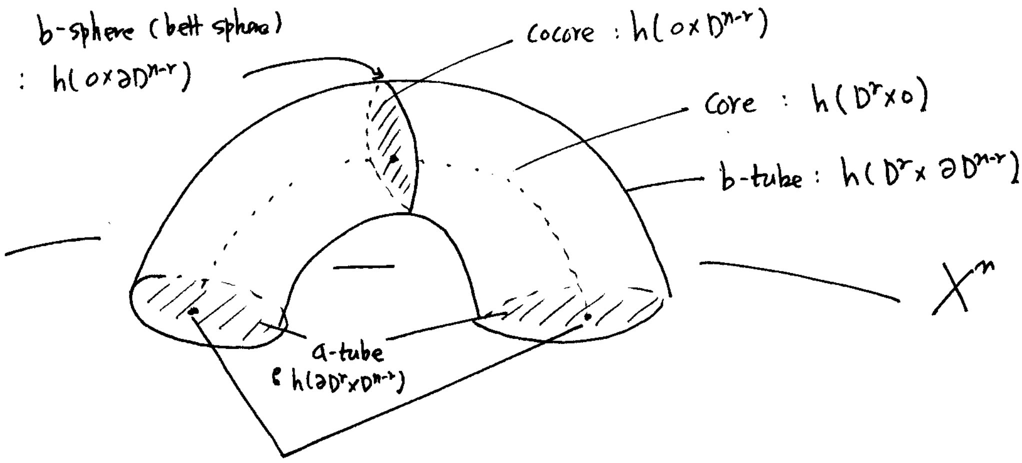
$\therefore$ $\exists$ c-homeo f_1'' for which $f_1'' h(m_1)$ coincides with m_1 as a point set and has the same orientation. //

Handlebody:

A handle of index r attached to the boundary of an n-manifold X, written as $X \underset{\varphi}{\cup} H^r$, consists of an embedding

$$\varphi : \partial D^r \times D^{n-r} \longrightarrow \partial X \qquad (h : D^r \times D^{n-r} \xrightarrow{\cong} H^r \ \text{s.t.}\ h|_{\partial D^r \times D^{n-r}} = \varphi)$$

and an associated identification space of $D^r \times D^{n-r}$ with X.



a-sphere (attaching sphere) $:= h(\partial D^r \times 0)$

$X \underset{\varphi}{\cup} H^r$ is specified by two pieces of data.

1) an embedding $\varphi_0 : S^{k-1} \longrightarrow \partial X$ with trivial normal bundle
2) a (normal) framing f of $\varphi_0(S^{k-1})$, or an identification of the normal bundle $\nu(\varphi_0(S^{k-1}))$ with $S^{k-1} \times \mathbb{R}^{n-k}$.

isotopy from (φ_0, f) to (φ_0', f') determines (up to isotopy) a diffeom. between $X \underset{\varphi}{\cup} H^r$ and $X \underset{\varphi'}{\cup} H^r$.

<u>Recall</u> (Whitney Thm): Every k-dim'l manifold embeds in $\mathbb{R}^{2k+1}$.

(Whitney immersion Thm) Every k-dim'l mfd X may be immersed in $\mathbb{R}^{2k}$.

$\therefore$ $2(l+1) \leq m \Rightarrow$ any two homotopic embeddings $N^l \hookrightarrow M^m$ is always isotopic.

① <u>Attaching k-handle on n-manifold.</u>

if $2k \leqq n-1$, then any homotopic embedding $S^{k-1} \hookrightarrow \partial X$ is isotopic.

$\therefore$ $X^n \cup$ k-handles, $(2k \leq n-1)$, is determined by X,

$\#$(k-handles) and framings of the k-handles.

ex) 1-handle in 3-mfd, 1-handle in 4-mfd. $\cdots$

② <u>Framings on a sphere S^{k-1} in ∂X^n.</u>

$\{$isotopy class of framings of S^{k-1} in $\partial X^n\}$ $\xleftrightarrow{\text{bij}}$ $\pi_{k-1}(O(n-k))$

· $\pi_0(O(n-1)) \cong \mathbb{Z}_2$ for $n \geqq 2$. $\Rightarrow$ 2 framings are possible (orientation)

· $(n-1)$ handles $(n \neq 2)$ and n-handles in general.

$\pi_{n-2}(O(1)) = \pi_{n-1}(O(0)) = 0 \Rightarrow \exists$ unique framing.

· For $n \leqq 4$, any self diffeom of S^{n-1} is isotopic to identity or a reflection.

$\therefore$ $\exists$ unique way to attach an n-handle to an S^{n-1} boundary component.

- Attaching 2-handle

$$\pi_1(O(n-2)) \cong \begin{cases} \mathbb{Z} & \text{if } n=4 \\ \mathbb{Z}_2 & \text{if } n>4 \end{cases}$$

Def. X: compact n-manifold with $\partial X = \partial_- X \amalg \partial_+ X$

(If X is oriented, then orient ∂X such that $\partial X = \overline{\partial_- X} \amalg \partial_+ X$.)

A handle decomposition of X relative to $\partial_- X$ is

$$X = (I \times \partial_- X) \cup \text{handles}.$$

Recall 1. Every smooth compact manifold pair $(X, \partial_- X)$ admits a handle decomp.

($\because$ Any smooth function $f : X \longrightarrow [0,1]$ with $f^{-1}(0) = \partial_- X$, $f^{-1}(1) = \partial_+ X$ can be perturbed into a Morse function with no critical points on ∂X.)

2. (Rourke - Sanderson) Any PL-pair admits a PL handle decomposition constructed from a triangulation.

3. Moise ($n=3$), Kirby - Siebenman ($n \geq 6$), Freedman - Quinn ($n=5$):

A topological manifold pair $(X, \partial X)$ with $\dim X \neq 4$ always admits a topological handle decomposition. (w/ attaching maps are homeo. embeddings)

4. $(X^4, \partial_- X)$ admits a topological handle decomposition iff X^4 is smoothable.

($\because$ Moise: Any homeomorphic embedding of smooth 3-mfd is uniquely smoothable. $\therefore$ the attaching maps can always be smoothed by an isotopy.)

Modification of Handle decomposition

1. Reordering : $X' = X \cup H^{(r)} \cup H^{(s)}$ with $s \leq r$

$\Rightarrow$ $X' \cong X \cup H^{(s)} \cup H^{(r)}$ with $H^{(r)}$ and $H^{(s)}$ disjoint.

($\because$ dim (b-sphere of $H^{(r)}$) + dim (a-sphere of $H^{(s)}$) $= n-r-1 + s-1 = n+(s-r)-2 < n-1$. $\Rightarrow$ we can make them disjoint)

2. Cancellation : $X' = X \cup H^{(r)} \cup H^{(r+1)}$ with $H^{(r)}$ and $H^{(r+1)}$ complementary

(i.e. the b-sphere of $H^{(r)}$ and the a-sphere of $H^{(r+1)}$ intersect transversally in just one point).

Then $X' \cong X$.

Introduction : $X' = X \cup D^n$, $D^n \cap X = D^n \cap \partial_+ X =$ face B_1 of D^n

$\Rightarrow X \cong X' = X \cup H^{(r)} \cup H^{(r+1)}$ with $H^{(r)}$ and $H^{(r+1)}$ complementary.

3. Handle Slide : $H_1^{(k)}, H_2^{(k)}$; $1 \leq k < n$, attached on ∂X.

A handle slide H_1 over H_2 is given by

- $A_{H_1} \subset \partial(X \cup H_2)$

  isotope A_{H_1} through $\partial(X \cup H_2)$ to B_{H_2}.

- in the intermediate step, A_{H_1} meets B_{H_2} at one point p

$$\underbrace{\underset{\dim = k-1}{T_p A_{H_1}} \oplus \underset{\dim = n-k-1}{T_p B_{H_2}}}_{\text{sum} = n-2} \quad \text{has codim } 1 \text{ in } T_p(\partial(X \cup H_2))$$

$\therefore$ chose a direction pushing A_{H_1} off of B_{H_2}.

4. Dual handle decomposition.

$$X^n = (\partial_- X \times I) \cup H_1 \cup H_2 \cup \cdots \cup H_k \cup (\partial_+ X \times I)$$

Let $X_{i-1} = (\partial_- X \times I) \cup H_1 \cup \cdots \cup H_{i-1}$.

and let $X_{i+1}^c = H_{i+1} \cup H_{i+2} \cup \cdots \cup H_k \cup (\partial_+ X \times I)$

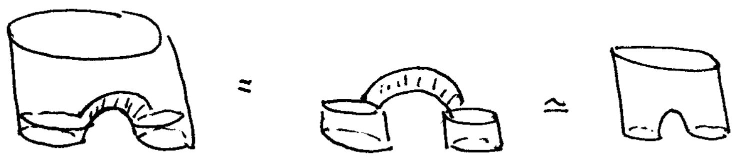
Then H_i can be regarded as a handle H_i^* on X_{i+1}^c with attaching map $h_i^* = h_i \circ t$ where $t: D^p \times D^q \longrightarrow D^q \times D^p$.

(Rmk : $\text{index}(H_i^*) = n - \text{index}(H_i)$)

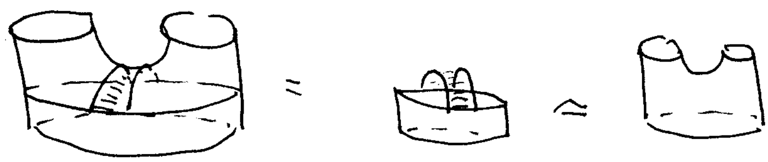
and $X^n = (\partial_+ X \times I) \cup H_k^* \cup H_{k-1}^* \cup \cdots \cup H_2^* \cup H_1^* \cup (\partial_- X \times I)$

: dual handle decomposition.

ex)



Upside down.



Observe that the a-sphere & b-sphere is interchanged in the two figure.

5. Trading handle.

$$\partial(M \cup_f H^r) \approx \partial(M \cup H^{n-r-1})$$
$$\| \qquad\qquad \|$$
$$\partial M \# S^r \times S^{n-r-1}$$

Remark : Surgery.

Convention: $S^{-1} = \partial D^0 = \phi$

$\varphi : S^k \hookrightarrow M^m \quad (-1 \le k < n)$ with framing f on $\varphi(S^k)$

$\Rightarrow (\varphi, f)$ determines an embedding $\hat{\varphi} : S^k \times D^{m-k} \hookrightarrow M$

up to isotopy.

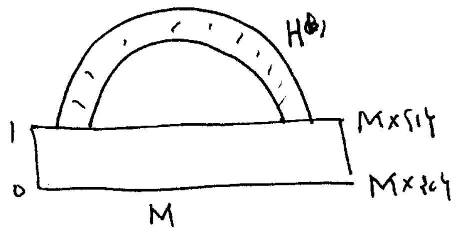
Surgery on $(\varphi, f) = (M^m \setminus \hat{\varphi}(S^k \times D^{n-k})) \underset{\hat{\varphi}|_{S^k \times S^{n-k-1}}}{\cup} (D^{k+1} \times S^{m-k-1})$

① k-handle attachment on $(X, \partial X) \longleftrightarrow (k-1)$ surgery on $\partial_+ X$

$(\because$ Consider $X^n = M^m \times \underset{[0,1]}{\overset{\shortparallel}{I}}$. and $D^k \times D^{m-k+1}$ by attaching

$S^{k-1} \times D^{m-k+1}$ to its image under $\varphi \times \{1\}$.

Then $\partial_+ (X^n \underset{(\varphi, f)}{\cup} H^{(k)}) = (M^m \setminus \hat{\varphi}(S^{k-1} \times D^{m-k+1})) \underset{\hat{\varphi}}{\cup} (D^k \times S^{m-k})$ $)$.



② $D^{k+1} \times S^{n-k-1} = D^{k+1} \times (D_-^{n-k-1} \cup D_+^{n-k-1})$

$= (D^{k+1} \times D_-^{n-k-1}) \cup (D^{k+1} \times D_+^{n-k-1})$

: 0-handle $\cup$ $(n-k-1)$-handle

k-surgery $= (M^n \setminus \hat{\varphi}(S^k \times D^{n-k})) \cup (k+1)$ handle $\cup$ n-handle.

Recall (Kirby's Thm) $L, L' \subset S^3$: two integer framed links

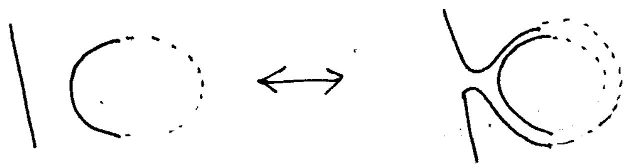
$S^3_L \cong S^3_{L'}$ iff $L \underset{\phi}{\sim} L'$

finite sequence of KI & KII moves

KI-move: $L \longleftrightarrow L \;\; O^{\pm 1}$

(In Ribbon notation: $L \underset{O^{+1}}{\infty} \leftrightarrow L \leftrightarrow L \underset{O^{-1}}{\infty}$)

KII-move: (= 2-handle slide in 4-mfd picture)



Remark ① $X^4_{L \cup O^1} \approx X^4_L \# \mathbb{C}P^2$

$X^4_{L \cup O^{-1}} \approx X^4_L \# \overline{\mathbb{C}P}^2$ $\Big) \not\approx X^4_L$.

But $\partial(X^4_L \cup O^1) = S^3_L \# S^3_{O^1} = S^3_L \# S^3 \cong S^3_L$.

② H_1, H_2: 2-handles attached along K_1 and K_2.

$K_2' =$ a parallel curve determining the framing of K_2.

$\Rightarrow K_2'$ bounds a disk $\underbrace{\varphi(D^2 \times pt)}$ $\subset \partial(X \cup H_2)$

↳ so called "core"

Slide H_1 (or attaching sphere K_1) by isotoping K_1 over one such disk, $K_1 \rightsquigarrow K_1 \#_b K_2'$.

(Any band disjoint from the rest of link are allowed)

If 2-handles are attached to D^4 along the oriented framed link $L = K_1 \cup \cdots \cup K_n$.

Choose a basis $\alpha_i \in H_2(X)$ s.t. $Q_X = (lk(K_i, K_j))$

If we slide H_i over H_j, then we change the basis

$$\alpha_i \longmapsto \alpha_i \pm \alpha_j.$$

$$\therefore \quad (\alpha_i \pm \alpha_j)^2 = \alpha_i^2 \pm 2\alpha_i \cdot \alpha_j + \alpha_j^2$$

$$= n_i \pm 2\, lk(K_i, K_j) + n_j$$

$$\therefore \text{ new framing} = n_i + n_j \pm 2\, lk(K_i, K_j)$$

depends on the band sum. //

③ Since KII preserves the smooth type of the 4-mfd, it does not change the homeo type of the boundary 3-mfd. //

From ① & ③ ⇐) of the Kirby's Thm is clear. //

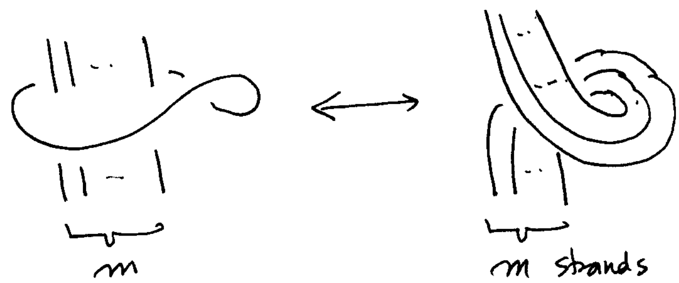
$\{$closed connected oriented 3-mfds$\}$ / homeomorphism

$=$ $\{$(unoriented) framed links in $S^3\}$ / isotopy, KI, KII.

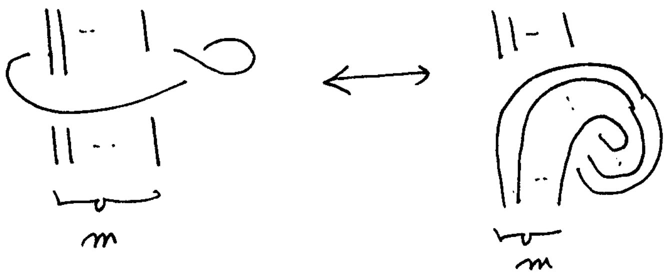
$\{$Knots$\}$ / isotopy of $\mathbb{R}^3$ $=$ $\{$Knot diagram$\}$ / RI, RII, RIII & isotopy of $\mathbb{R}^2$

<u>Modification of the Kirby moves</u> (the Fenn-Rourke moves)

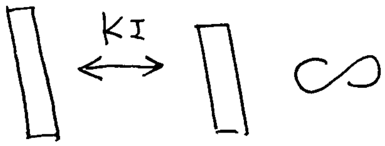
FR$_+$ move



FR$_-$ move

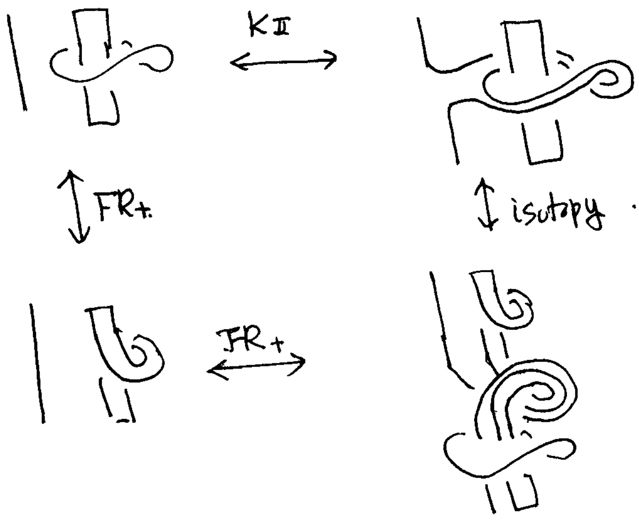


<u>Remark</u>: 1.



(KI move is a FR move with 0 strand.)

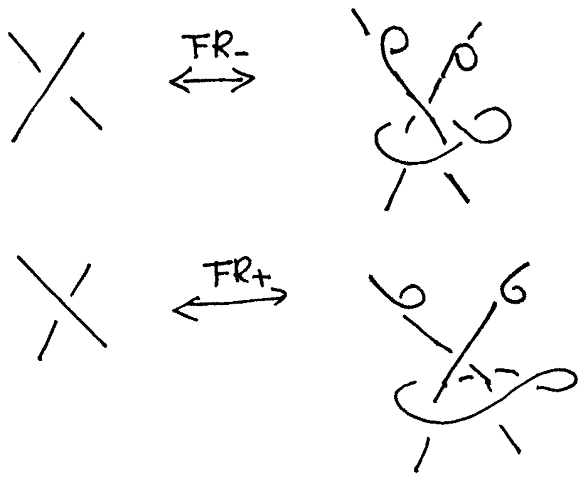
2.



∴ KII move over an unknot (with framing ±1) can be realized as a composition of $FR_\pm$ and isotopy.

3. Any knot can be unknotted by using $FR_\pm$.

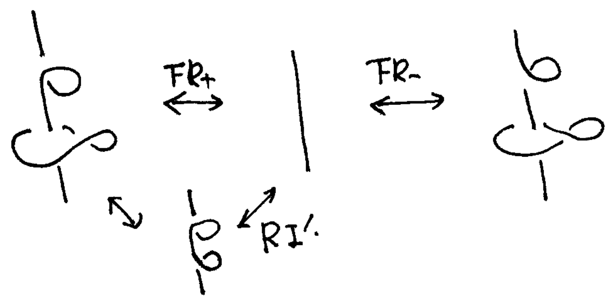
(∵



)

4. We can make the framing of the unknot to ±1 by using FR-moves

(∵



)

∴ KII moves can be realized as a composition of $FR_\pm$'s and isotopies.

$$\boxed{\partial X_{L_1} \cong \partial X_{L_2} \quad \Rightarrow \quad L_1 \sim_0 L_2}$$

$N^4 = X_{L_1} \cup (\partial X_{L_1} \times [1,2]) \cup \overline{X_{L_2}}$: closed oriented 4-mfd

<u>Remark</u> :

$$Q_{X^4 \# \mathbb{C}P^2} = \left[\begin{array}{c|c} Q_{X^4} & 0 \\ \hline 0 & 1 \end{array}\right] \qquad \therefore \quad \sigma(X \# \mathbb{C}P^2) = \sigma(X)+1$$

$$Q_{X^4 \# \overline{\mathbb{C}P^2}} = \left[\begin{array}{c|c} Q_{X^4} & 0 \\ \hline 0 & -1 \end{array}\right] \qquad \sigma(X \# \overline{\mathbb{C}P^2}) = \sigma(X)-1 .$$

Change $L_1 \rightsquigarrow \begin{cases} L_1 \amalg |\sigma(N)|\, \mathbb{C}P^2 & \text{if } \sigma(N)<0 \\ L_1 \amalg |\sigma(N)|\, \overline{\mathbb{C}P^2} & \text{if } \sigma(N)>0 \end{cases}$

i.e. $L_1' = \begin{cases} L_1 \amalg |\sigma(N)|\, O^{+1} \\ \text{or} \\ L_1 \amalg |\sigma(N)|\, O^{-1}. \end{cases}$ (we call L_1' as L_1)

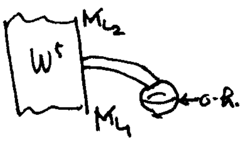
$\Rightarrow$ $N'^4 = X_{L_1'} \cup (\partial X_{L_1'} \times [1,2]) \cup \overline{X_{L_2}}$ has $\sigma(N'^4) = 0$.

$\Rightarrow$ N' bounds an oriented connected 5-mfd W^5.

Let $f : W^5 \longrightarrow [1,2]$: Morse function for which $f^{-1}(1) = X_{L_1}$
$f^{-1}(2) = X_{L_2}$.

and $f|_{\partial X_{L_1} \times [1,2]} = pr_2$.

$W^5 = (X_{L_1} \times [0,1]) \cup$ 0-hs $\cup$ 1-hs $\cup$ 2-hs $\cup$ 3-hs $\cup$ 4-hs $\cup$ 5-hs $\cup (X_{L_2} \times [0,1])$

① 

W^5 : connected

If $\exists$ 0-h, then it should be connected to $X_L \times [0,1]$ by 1-h.

$\therefore$ we can delete 0-h/1-h pair.

$\therefore$ We may assume there is no 0-hs

(in the same arguement to the upside down picture, we may assume there is no 5-hs ")

② $\partial_+((X_{L_1} \times [0,1]) \cup H^{(1)}) \cong \partial_+((X_{L_1} \times [0,1]) \cup H^{(3)})$

$$\begin{matrix} \| & & /\!/ \\ & X_{L_1} \# S^1 \times S^3 & \end{matrix}$$

$\therefore$ Replace all critical points of index 1 $\rightsquigarrow$ index 3

index 4 $\rightsquigarrow$ index 2.

$\therefore$ ① & ② $\Rightarrow \exists$ W^5 (different from the above W^5, but w/ the same bdry N) Morse function $f : W^5 \longrightarrow [1,2]$ with critical points of index 2 & 3 only. (by rearranging critical pts, we may assume that all critical points of index 2 belonging to $f^{-1}([1, \frac{3}{2}])$

& 3 " $f^{-1}([\frac{3}{2}, 2])$

$\therefore$ $f^{-1}(\frac{3}{2}) \cong X_{L_1} \# r\, S^2 \times S^2 \# s\, S^2 \tilde{\times} S^2 = X_{L_1'}$

$\cong X_{L_2} \# r'\, S^2 \times S^2 \# s'\, S^2 \tilde{\times} S^2 = X_{L_2'}$

$X_{L_1'} \cong X_{L_2'} \overset{\text{Let}}{=} X^4$.

$L_1 \xrightarrow[KI, KII]{} L_1'$

$L_2 \xrightarrow[KI, KII]{} L_2'$

Two different handlebody structure

$f_i : X \longrightarrow [-1, 1]$, $i = 1, 2$, $f_i^{-1}(-1) =$ the only critical point of index 0

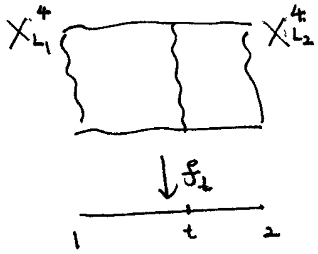
$f_i^{-1}(0) \cong S^3$.

$f_i^{-1}(1) = \partial X = \partial X_{L_i}$

& all critical points of index 2 in $f^{-1}(0,1)$.

∃ homotopy $f_t : X' \to [-1,1]$, $t \in [1,2]$ s.t. each f_t is a Morse ftn except for a finite number of t.

<u>Cerf theory</u>



$X_4 \cong X_{L_2}$

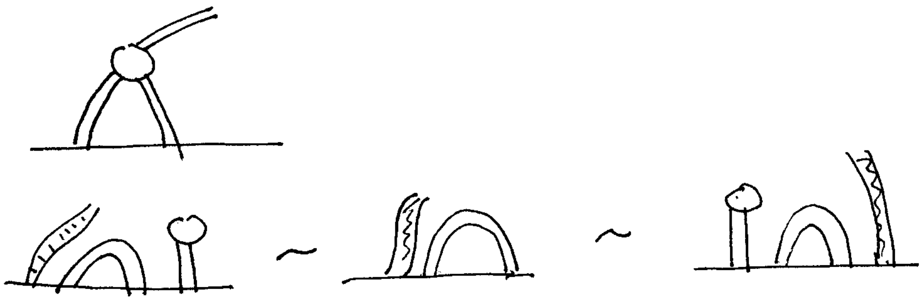
$\Rightarrow$ $L_1 \rightsquigarrow L_2$

- Birth/deaths of complementary handle pair.
- handle slide
- isotopy

① <u>eliminate the 0-handles</u>

- 0-h attached to the base of the cobordism by a 1-handle.
- There is a choice in the 0-h/1-h pair cancellation procedure, but the choices are related by 1-h slides

(ex)



)

We may assume: f_t introduces no 0-handles and no 4-handles

② births / deaths take place in a ball

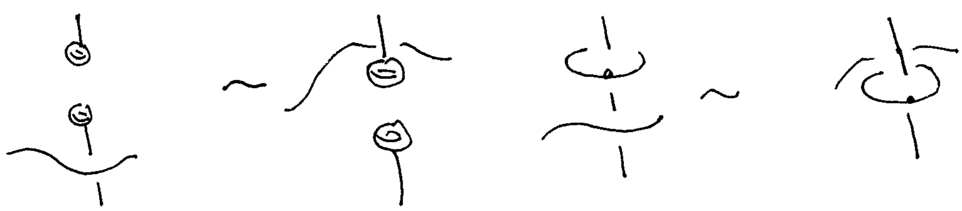
$\Rightarrow$ We may assume they can be pictured as taking place in the bottom of the cobordism

③ $L_1 \rightsquigarrow L_2$

1) 1-h/2-h pair birth/death.

.

2) 2-h slide under the 1-h.



3) 2-handle slide → KII-moves ~~at~~ some level on the cobordism.

explain

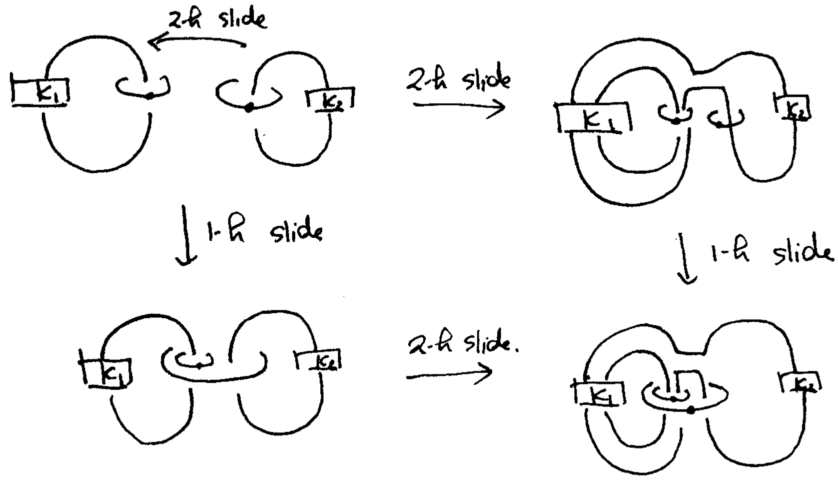
<u>WTS</u>: 'birth/death of 1-h/2-h pair and the 1-handle slide'

by using KI-move

a) births and deaths may be reordered so that all births takes place before any deaths.

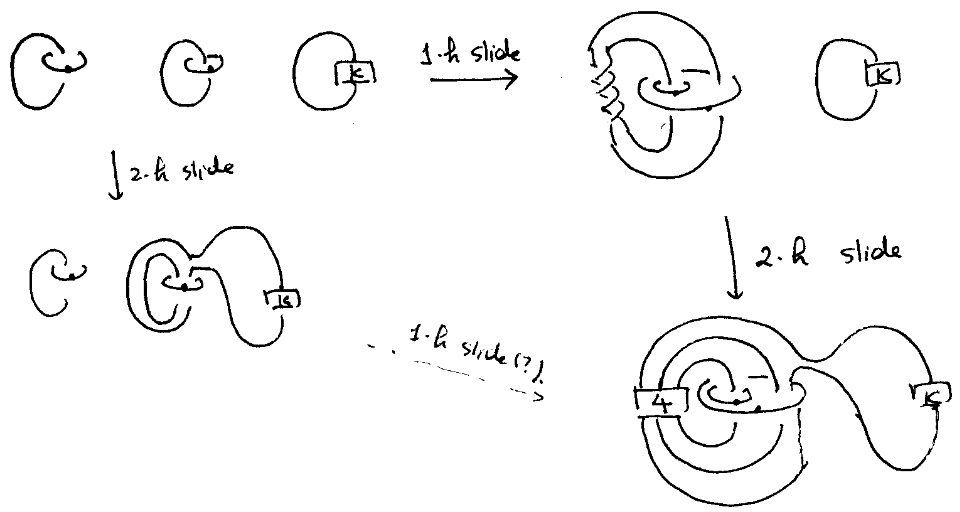
b) 2-h slide → 1-h slide ⇒ 1-h slide → 2-h slide.

a))



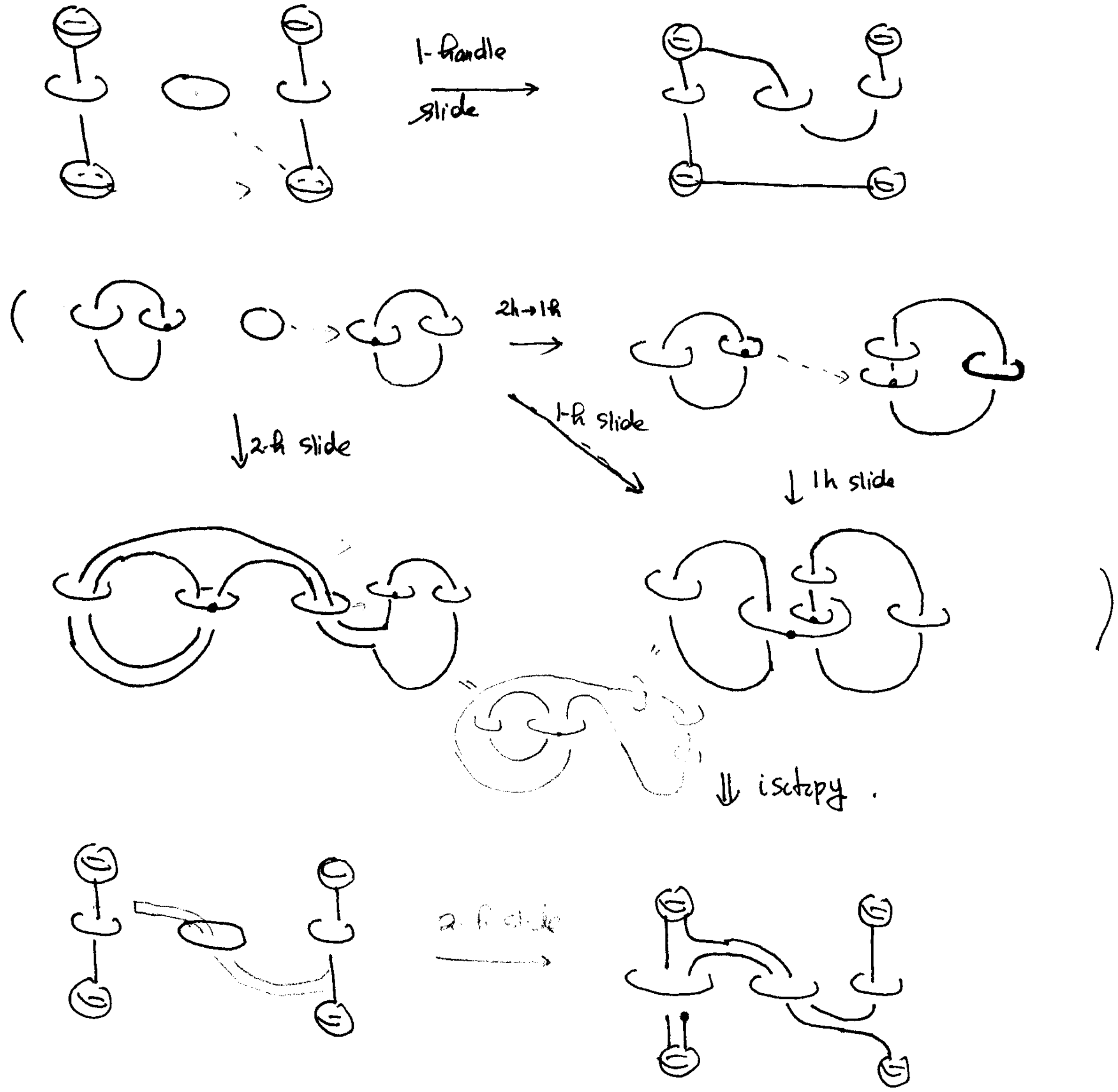
cf) the reverse of '1-h slide → 2-handle slide" has no meaning.

(∵



∴ We may assume that the 1-handle slides all proceed the 2-handle slide.

c) exchange the 1-handle slides for 2-handle slides



· 1-handle simple if its cancelling 2-handles does not go round any other 1-handle.

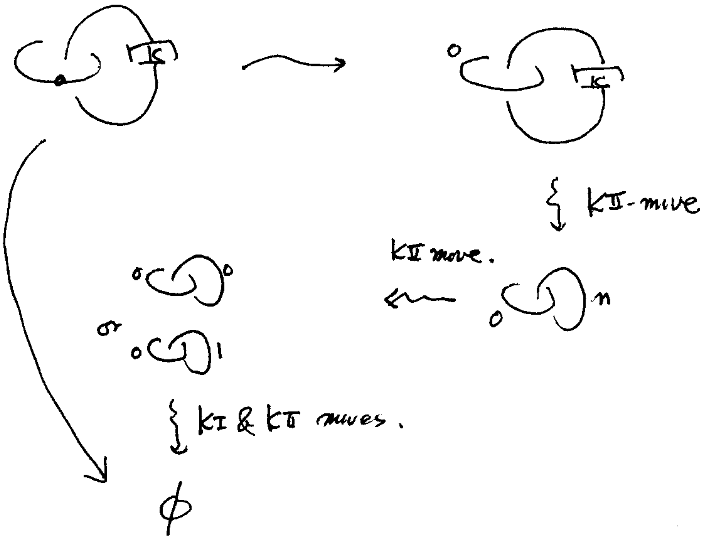
· Call a 1-handle slide simple if the passing 1-handle is simple.

The above procedure reduce one 1-handle slide by exchanging it a 2-handle slide.

Repeat it: all 1-h slides are exchanged by 2-h slides.

c) Trade in the 1-handles for 2-handles

$$\partial(X \cup H^{(1)}) = \partial X \# S^1 \times S^2 = \partial(X \cup \underbrace{\text{0-framed 2-handle}}_{\text{cancelling pair of the 1-handle}})$$



Prop'n. M: n-mfd. w/ $n \geq 4$
(need not be compact)

C : null homotopic circle embedded in M

$\Rightarrow$ Surgery on C gives $M \# S$, where S : one of S^{n-2} bdle over S^2

pf) $M = M \# S^n$.

$C_0 \subset M \# S^n$ be the circle $\partial D^2 \times 0 \subset \partial(D^2 \times D^{n-1}) \cong S^n$.

$C \sim 0$ in $M \Rightarrow C \sim C_0$ $\therefore$ we may assume $C = C_0$

($\because$ For $2(l+1) \leq m$, two homotopic embedding $N^l \hookrightarrow M^m$
is always isotopic)

Doing surgery on $C_0 \Rightarrow M \# S$.

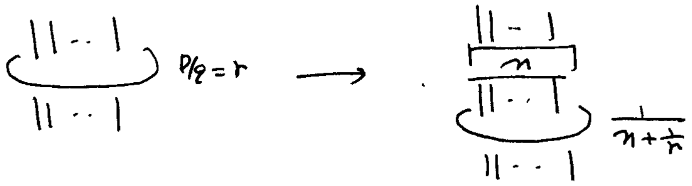
where $S = \partial(D^{n+1} \# 2\text{-}h)$

and $D^{n+1} \cup 2\text{-}h \cong D^{n-1}$ bundle over S^2.

$\therefore$ $\partial(D^{n+1} \cup 2\text{-}h) = S^{n-2}$ bundle over S^2

and which are classified by $\pi_1(O(n-1)) \cong \mathbb{Z}_2$. if $n \geq 4$. //)

n Rolfsen twist



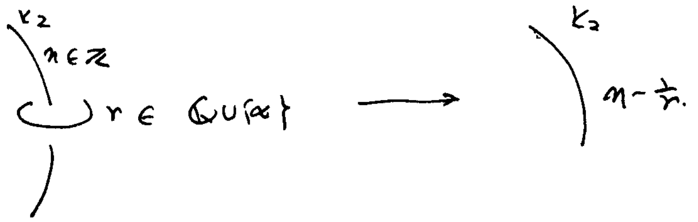
For each component K_i intersecting D,

$$r_i \longmapsto r_i + n(lk(K, K_i))^2.$$

ex) $\bigcirc^{p/q} \cong \bigcirc^{\frac{1}{n+\frac{q}{p}}} = \frac{p}{np+q}.$

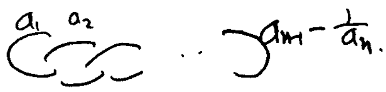
$$L(p, q) \cong L(p, np+q)$$

Slam dunk (T. Cochran)



$$\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cdots \cfrac{1}{a_{n-1} - \frac{1}{a_n}}}, \qquad a_i \in \mathbb{Z},$$

$X =$ 

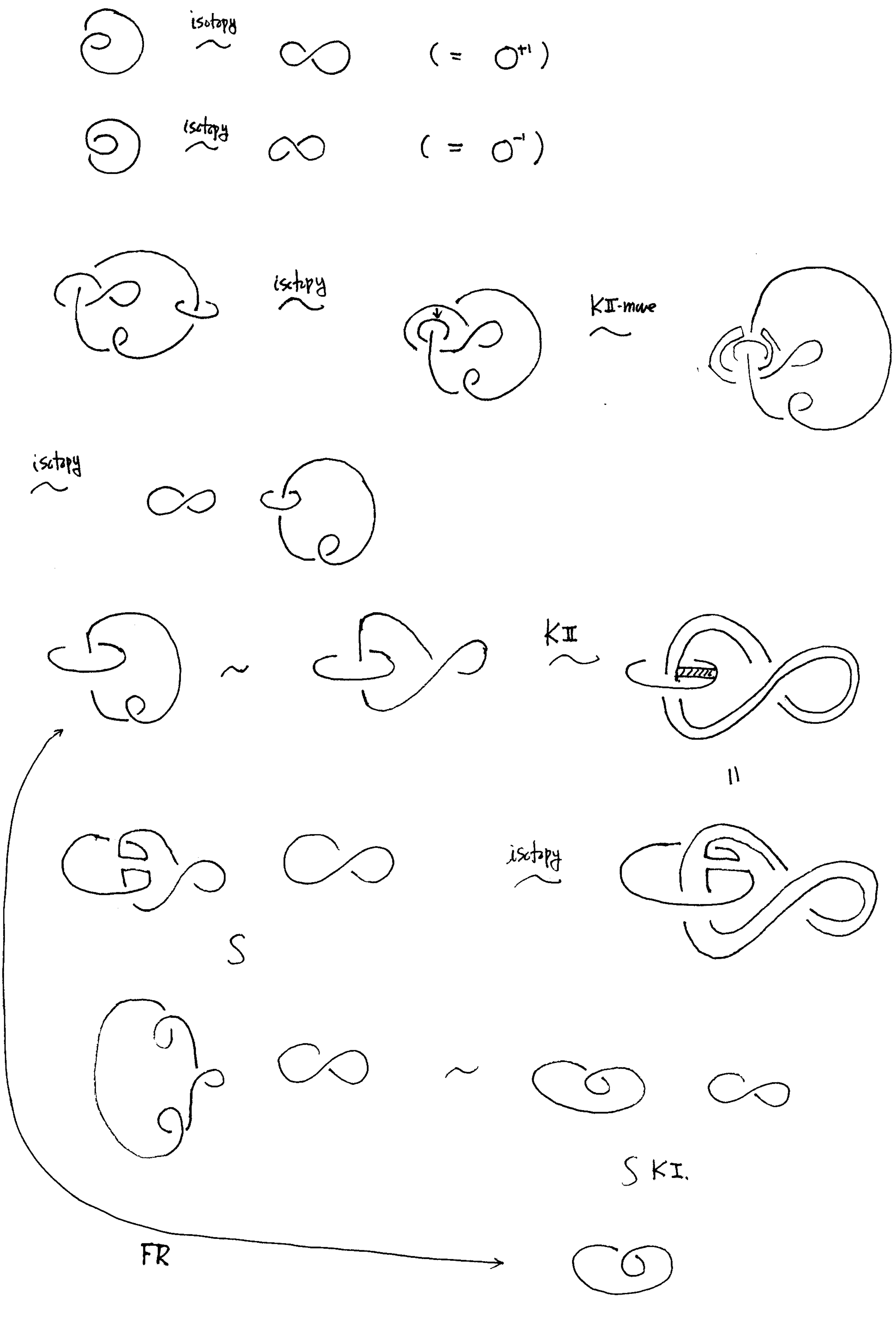
$\longrightarrow$ 

$\longrightarrow$ 

$\vdots$

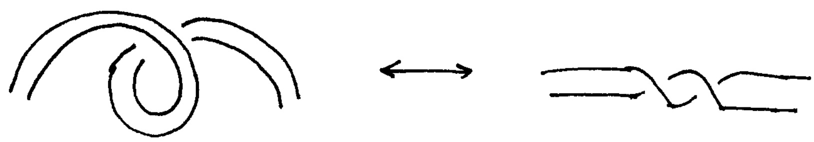
$\longrightarrow \bigcirc^{a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots \cfrac{1}{a_{n-1} - \frac{1}{a_n}}}}} = p/q.$ //

Some examples



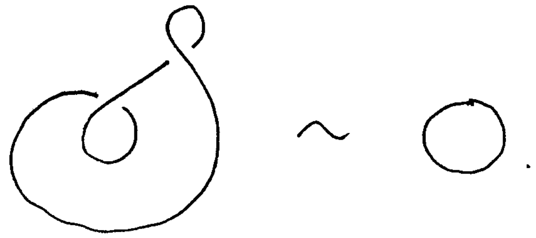
Some tricks

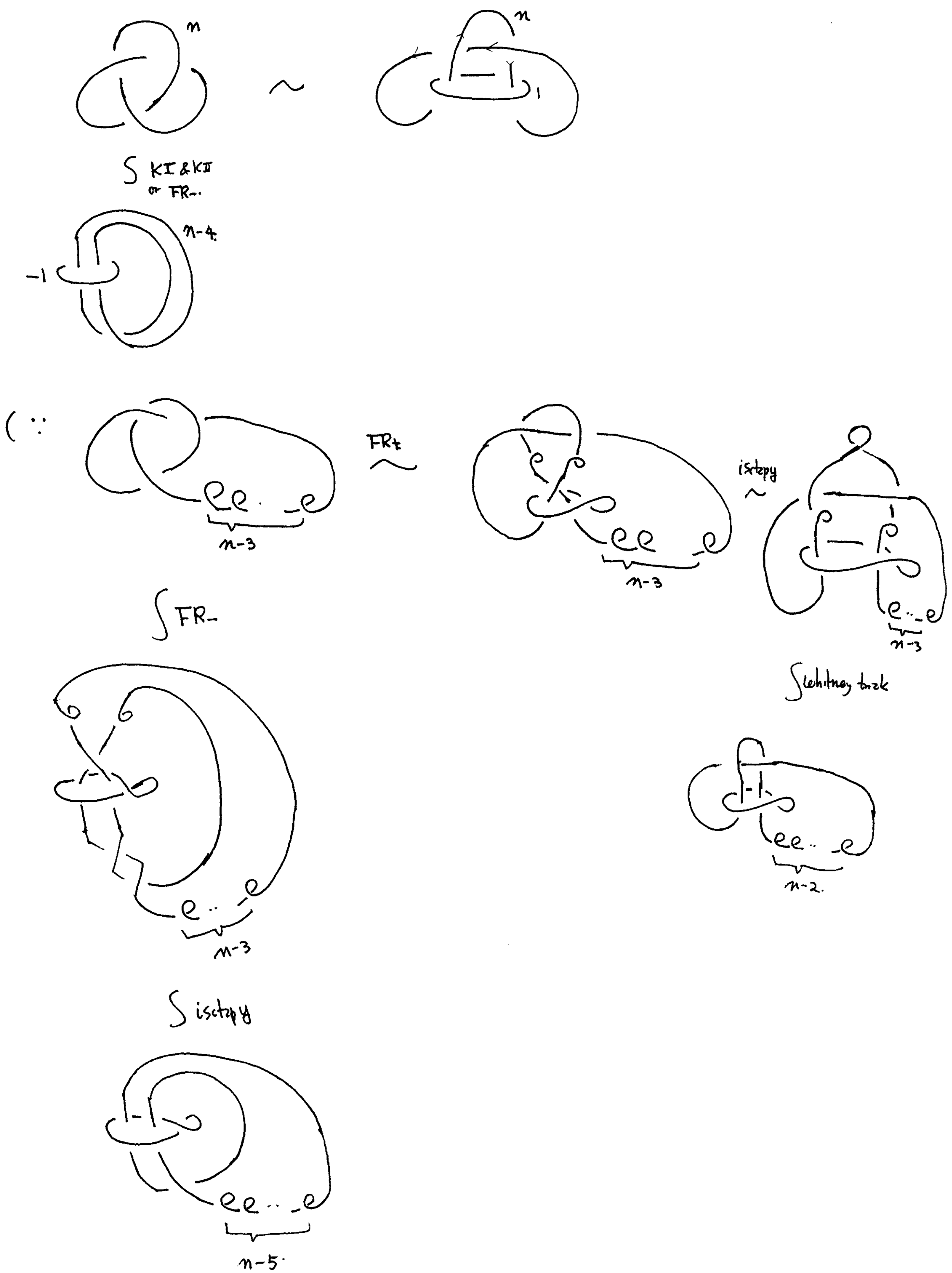
① Belt trick.



② Whitney trick

∴





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KI & KII
or FR-
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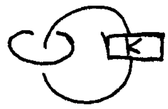
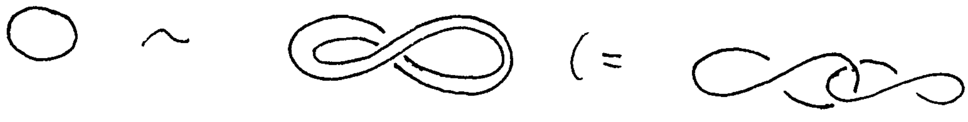
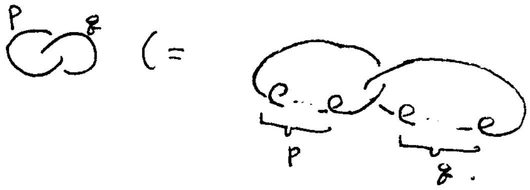
S^3 : ϕ ~ 

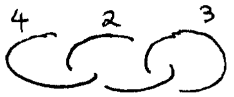
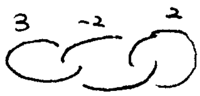
~  ......

$S^1 \times S^2$: 

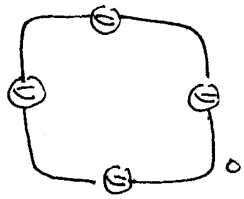
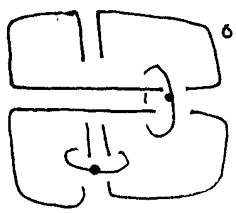
~  ...

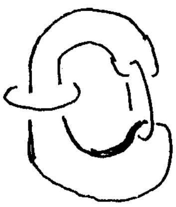
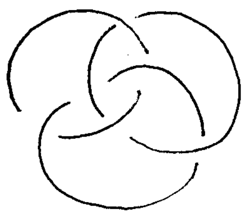
$L(pq-1, p)$: 

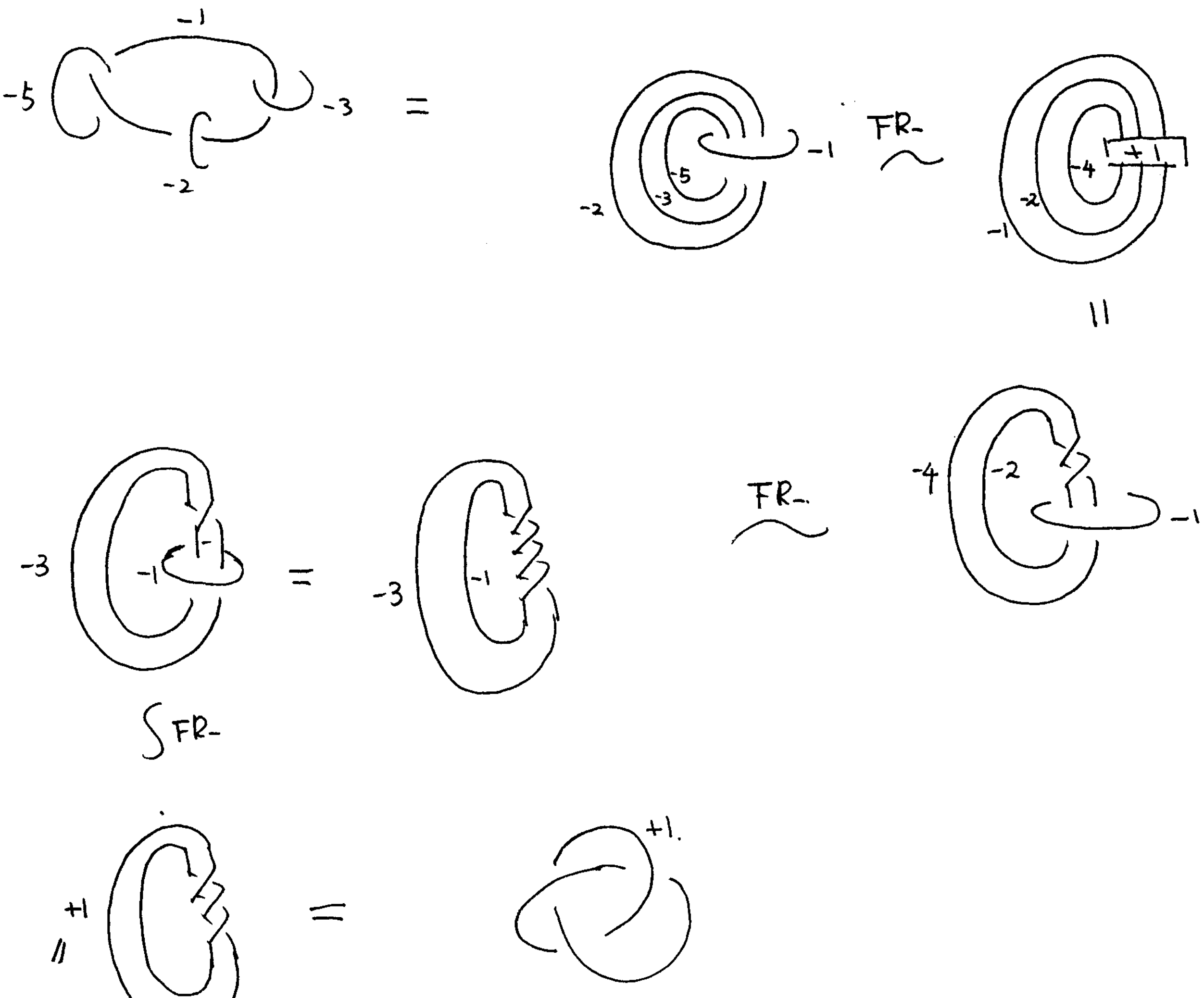
(Some examples by using Rolfsen's twist. (slam dunk $n \in \mathbb{Z}$, $r \in \mathbb{Q} \cup \{\infty\}$ ~ $n - \frac{1}{r}$)

 ~  $(= L(17, 5))$

$$\frac{17}{5} = 4 - \frac{1}{2 - \frac{1}{3}} \qquad \frac{17}{5} = 3 + \frac{1}{\frac{5}{2}} = 3 + \frac{1}{2 + \frac{1}{2}}.$$

T^3 $(= \partial(T^2 \times D^2))$ $T^2 \times D^2$:  = 

$\therefore$ T^3 :  = 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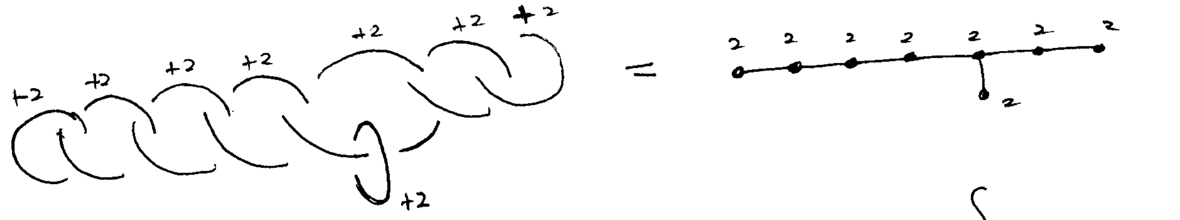
$$-3+(lk)^2 = -3+4=1$$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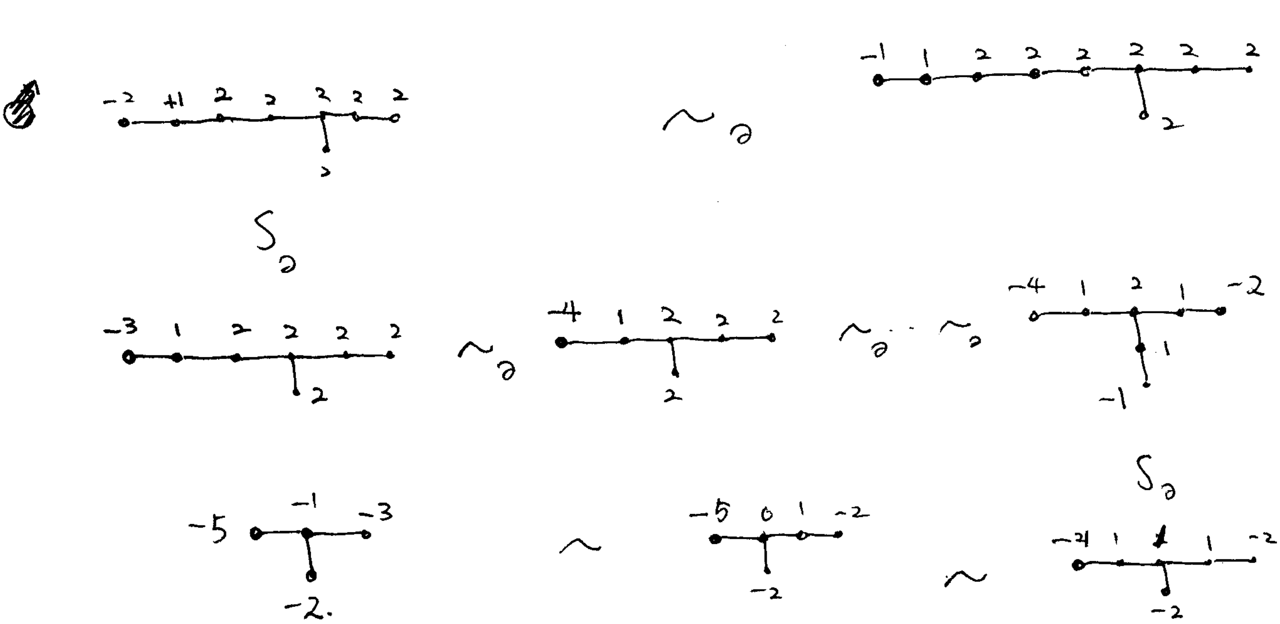
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-2
-4
1
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-2
-2