Introduction to finite type knot invariants

22 October 2004

What is a finite type invariant?

$$\mathcal{K} = \{\text{knots}\}$$

$$\mathcal{K}^1 = \{\text{knots with } 1 \text{ double point}\}$$

$$\vdots$$

$$\mathcal{K}^m = \{\text{knots with } m \text{ double point}\}$$

Given an abelian knot invariant

$V : \mathcal{K} \to A$ for an abelian group A,

define $V : \mathcal{K}^1 \to A$ by

$$V(\doublepoint) = V(\overcrossing) - V(\undercrossing).$$

This ‘double point resolving’ inductively defines

$$V : \mathcal{K}^m \to A \qquad \text{for any } m > 0.$$

Definition V is of type m if $V|_{\mathcal{K}^{m+1}} = 0$.

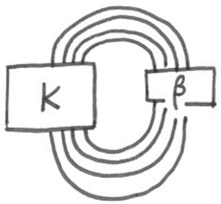
V is called of finite type (or Vassiliev) if it is of type m for some m.

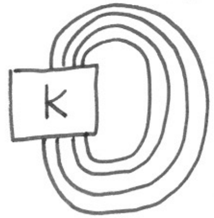
Finite type invariants are like generalized self-linking numbers.

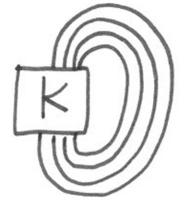
Consider the n-strand pure braid group P_n and its lower central series $P_n^{(0)} = P_n$, $P_n^{(m)} = [P_n^{(m-1)}, P_n]$.

Given a pure braid $\beta \in P_n$, the value of β in $P_n^{(0)}/P_n^{(1)}$ detects the linking numbers between strands. Suppose all the linking numbers are zero, then $\beta \in P_n^{(1)}$ and its image in $P_n^{(1)}/P_n^{(2)}$ may detect linkedness.

Theorem *If* V *is of type* m *and* $\beta \in P_n^{(m+1)}$*, then*

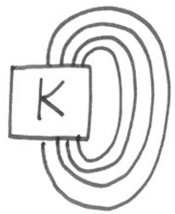
$V($ 

$) = V($ 

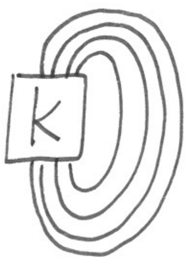
$)$ *for any knot* 

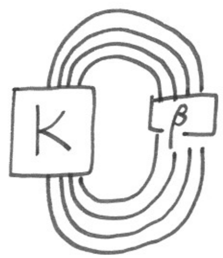
.

Theorem [T. B. Stanford (arXiv:math.GT/9805092)] *If* $V(K_1) = V(K_2)$ *for any* V *of type at most* m*, then for some*

$\beta \in P_n^{(m+1)}$ *and* 

,

$K_1 =$ 

and $K_2 =$ 

.

Finite type invariants are like polynomials.

The equation

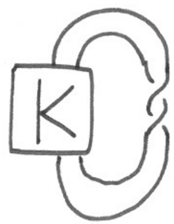
$$V(\times_{\bullet}) = V(\times_{+}) - V(\times_{-})$$

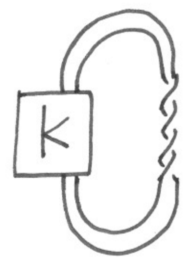
looks like

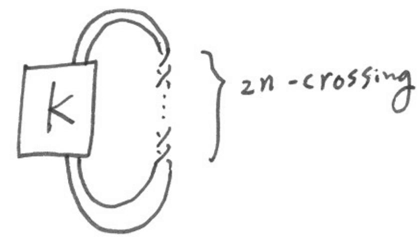
$\Delta f(x) = f(x+1) - f(x)$ for $f\colon \mathbb{Q} \longrightarrow \mathbb{Q}$. If $\Delta^{m+1} f = 0$, then f is a polynomial of degree m.

Likewise $V\big|_{\mathcal{K}^{m+1}} = 0$ implies V is of type m.

Theorem [M. Eisermann (2003) Trans. Amer. Math. Soc.] *Let $V : \mathcal{K} \longrightarrow \mathbb{Q}$ be a rational knot invariant. If $V(K_n)$ is a polynomial in n for any sequence of knot K_n and V is globally bounded by a polynomial of degree m in the crossing number,*

$K_1 =$ 

$K_2 =$ 

$K_n =$ 

Then V is of finite type.

The converse is also true.

Dominance over skein invariants

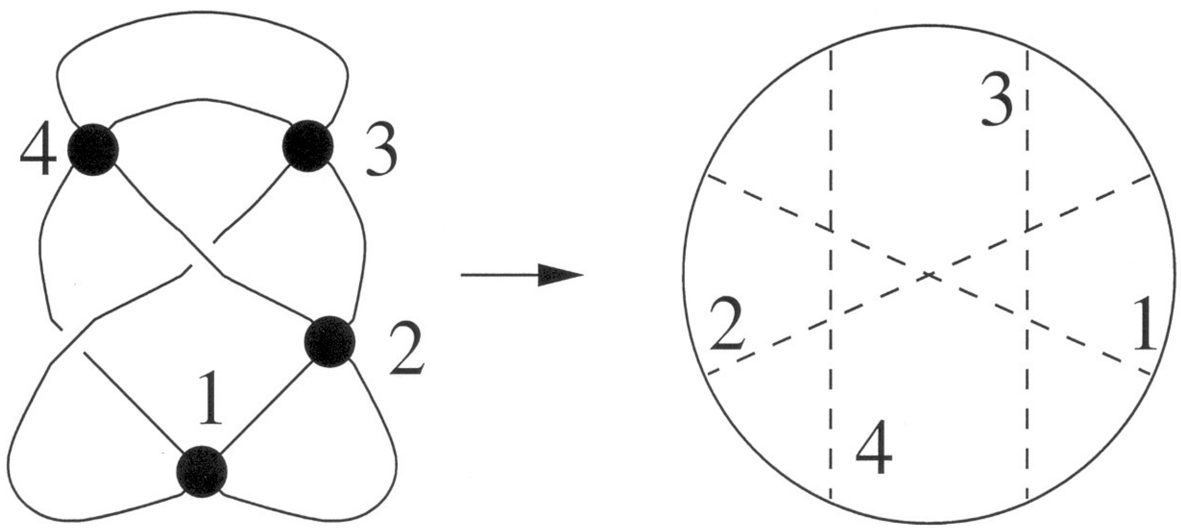
The set of finite type invariants is as strong as skein invariants including Alexander polynomial, Jones polynomial, Kauffman polynomial, homfly polynomial, and quantum invariants.
That is, if $V(K_1) = V(K_1)$ for every finite type invariant V then $P(K_1) = P(K_2)$ for any skein invariant P.

Theorem [J. S. Birman X.-S. Lin (1993) Invent. Math.] *Let* $J(K)(q)$ *be the Jones polynomial of a knot* K *(it is a Laurent polynomial in a variable* q*). Consider the power series expansion* $J(K)(e^x) = \sum_{m=0}^{\infty} V_m(K)x^m$*. Then each coefficient* $V_m(K)$ *is a finite type knot invariant. (And thus the Jones polynomial can be reconstructed from finite type information).*

Whether the signature of a knot can be expressed by the values of the finite type invariants is not known yet.

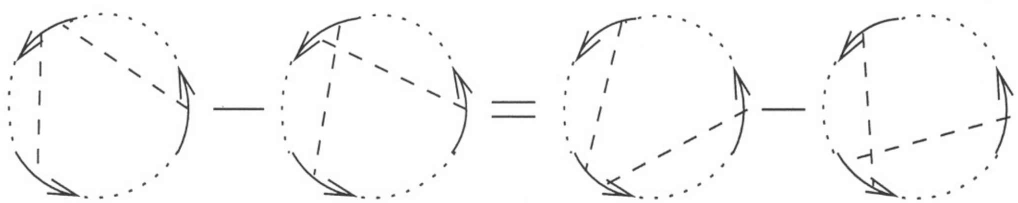
Chord diagram

Let V be of type m and $K \in \mathcal{K}^m$ be a singular knot with m double points. Since $V|_{\mathcal{K}^{m+1}} = 0$, V does not detect a crossing change of K, so that $V(K)$ depends only on the chord diagram of K.

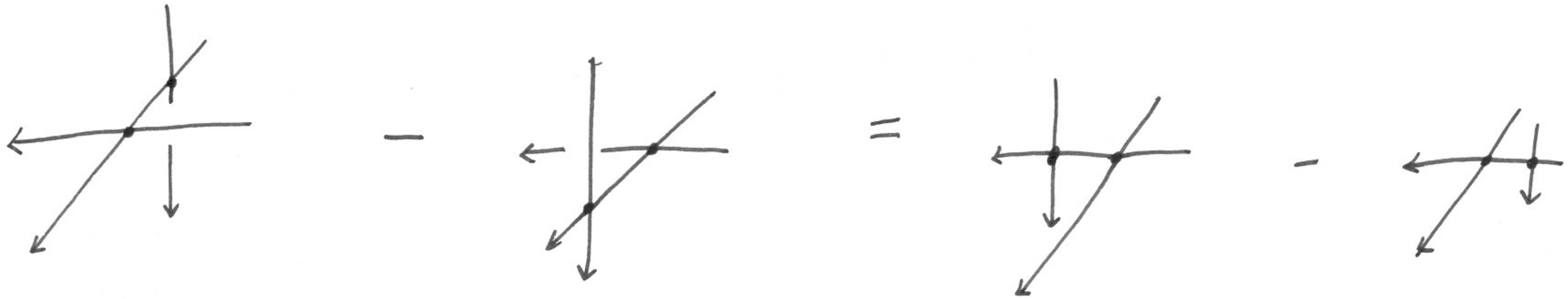


Definition Let $\mathcal{D}_m$ denote the space of all formal linear combinations with rational coefficients of m-chord diagrams. Let $\mathcal{A}^r_m$ be the quotient of $\mathcal{D}_m$ by all $4T$ and FI relations as drawn below. Let $\mathcal{A} := \bigoplus_m \mathcal{A}^r_m$.

$4T$:



FI :

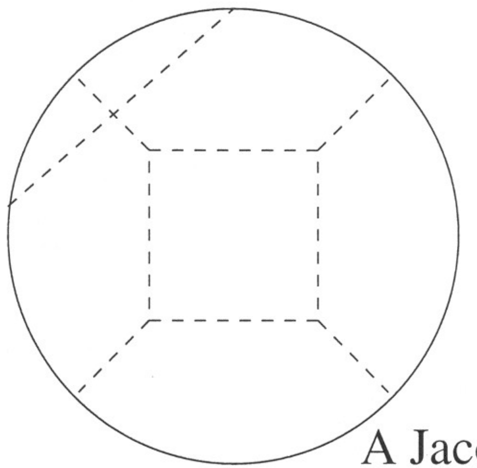


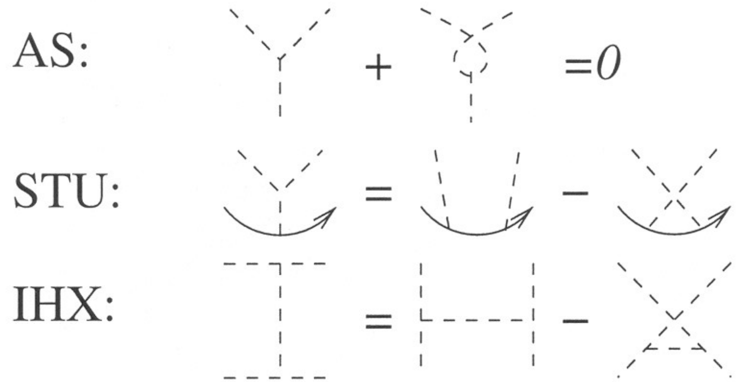
Theorem (The Fundamental Theorem) [M. Kontsevich (1993)]

- *(Easy part). If V is a rational valued type m invariant then $V^{(m)}$ defines a linear functional on $\mathcal{A}^r_m$. If in addition $V^{(m+1)} \equiv 0$, then V is of type m.*

- *(Hard part). For any linear functional W on $\mathcal{A}^r_m$ there is a rational valued type m invariant V so that $V^{(m)} = W$.*

Theorem [D. Bar-Natan (1995) Topology] *The algebra* $\mathcal{A}$ *is isomorphic to the algebra* $\mathcal{A}^t$ *generated by "Jacobi diagrams in a circle" (chord diagrams that are also allowed to have oriented internal trivalent vertices) modulo the* AS, STU *and* IHX *relations.*





A Jacobi diagram in a circle

$$AS : [x, y] + [y, x] = 0$$

$$STU : [x, y] = xy - yx$$

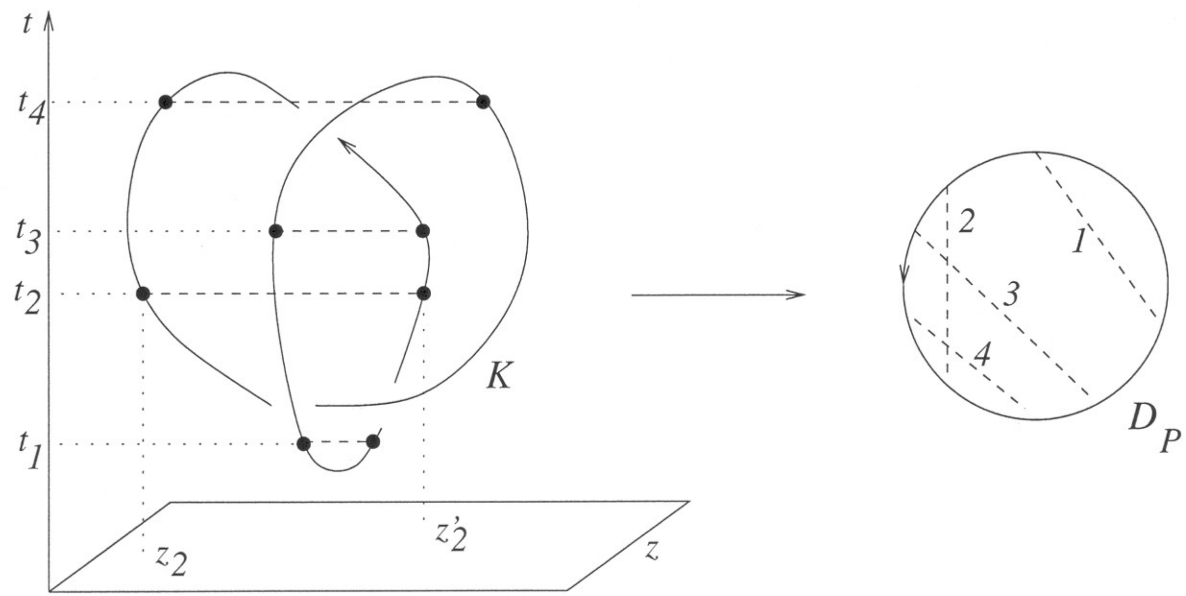
$$IHX : [[x, y], z] = [x, [y, z]] - [y, [x, z]]$$

Theorem [D. Bar-Natan (1995) Topology] *Given a finite dimensional metrized Lie algebra* $\mathfrak{g}$ *(e.g., any semi-simple Lie algebra) and a finite dimensional representation* R *of* $\mathfrak{g}$*, there is a linear functional* $W_{\mathfrak{g},R} \colon \mathcal{A} \to \mathbb{Q}$.

By the fundamental theorem, the above theorem implies that for each such $\mathfrak{g}$, R and $m > 0$, there exists a knot invariant of type m.

The Kontsevich integral

$$Z_1(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \sum\!\!\!\!\!\!\int_{\substack{t_1<\ldots<t_m \\ P=\{(z_i,z_i')\}}} (-1)^{\#P_\downarrow} D_P \bigwedge_{i=1}^{m} \frac{dz_i - dz_i'}{z_i - z_i'},$$



Definition $Z : \mathcal{K} \to \hat{\mathcal{A}}$ where $\hat{\mathcal{A}}$ is the graded completion of $\mathcal{A}$ is called a universal finite type invariant if for any $K \in \mathcal{K}^m$, $Z(K) = D_K + (\text{higher degree terms})$.

Theorem *The Kontsevich integral* Z_1 *is a universal finite type invariant.*

The fundamental theorem follows as a corollary.