※ $\mathfrak{g}$ is a Lie algebra if

⓪ $\mathfrak{g}$ is a vector space endowed with bilinear form

$$[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g}$$
$$(x,y) \longmapsto [x,y]$$

which is called bracket satisfying

① $[x,x]=0$, $\forall x\in\mathfrak{g}$

② $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$, $\forall x,y,z\in\mathfrak{g}$ [Jacobi's identity]

Example ① A : an associative algebra w/ multiplication $*$.

$\Rightarrow$ A has a Lie structure via $[x,y] := x*y - y*x$.

② V : a vector space, $A=\mathrm{End}(V)$ is an associative alg w/ $\circ$.

$\Rightarrow$ $\mathfrak{gl}(V) := (\mathrm{End}(V), [\cdot,\cdot])$: the general linear algebra

If $V=\mathbb{C}^n$, we write $\mathfrak{gl}_n$ or $\mathfrak{gl}(n,\mathbb{C})$, etc.

※ $\mathfrak{p}\subset\mathfrak{g}$ is called a Lie subalgebra if it is a subspace of $\mathfrak{g}$ and $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{p}$, i.e. $[x,y]\in\mathfrak{p}$, $\forall x,y\in\mathfrak{p}$

Example $\mathfrak{sl}(V)=\{x\in\mathfrak{gl}(V);\ tr(x)=0\}\subset\mathfrak{gl}(V)$

is a subspace clearly.

And $tr([u,v]) = tr(u\circ v - v\circ u) = 0$, $\forall u,v\in\mathfrak{gl}(V)$.

Moreover, $[\mathfrak{sl}(V),\mathfrak{sl}(V)]\subset\mathfrak{sl}(V)$.

※ A subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is called an ideal if

$[\mathfrak{a}, \mathfrak{g}] \subset \mathfrak{a}$, that is, $[a, x] \in \mathfrak{a}$ for all $a \in \mathfrak{a}$, $x \in \mathfrak{g}$.

Example $\mathfrak{sl}(V)$ is an ideal of $\mathfrak{gl}(V)$.

※ $\mathfrak{g}$ is called a simple Lie algebra if $\mathfrak{g}$ has only trivial ideals 0 and $\mathfrak{g}$ and $\mathfrak{g}$ is Not abelian, that is, $[x, y] \neq 0$ for some $x, y \in \mathfrak{g}$.

※ $\phi : \mathfrak{g} \to \mathfrak{g}'$ is a Lie alg homomorphism if

⓪ ϕ is a linear map,

① $\phi([x, y]) = [\phi(x), \phi(y)]$, $\forall\, x, y \in \mathfrak{g}$.

Example For a Lie algebra $\mathfrak{g}$, $\mathfrak{gl}(\mathfrak{g})$ is well-defined.

For $x \in \mathfrak{g}$, define

$$\begin{aligned} ad(x) : \mathfrak{g} &\longrightarrow \mathfrak{g} \quad \text{by} \\ y &\longmapsto [x, y]. \end{aligned}$$

Then $ad(x)$ is a linear map, i.e. $ad(x) \in \mathfrak{gl}(\mathfrak{g})$.

Define

$$\begin{aligned} ad : \mathfrak{g} &\longrightarrow \mathfrak{gl}(\mathfrak{g}) \quad \text{a (well-defined) map.} \\ x &\longmapsto ad(x) \end{aligned}$$

Then ad is a linear map also. And we have

$$ad([x, y]) = [ad(x), ad(y)]$$

since
$$\begin{cases} ad([x,y])(z) = [[x,y],z] = -[z,[x,y]] = [x,[y,z]] + [y,[z,x]] \\ [ad\,x, ad\,y](z) = ad\,x \circ ad\,y(z) - ad\,y \circ ad\,x(z) = [x,[y,z]] - [y,[x,z]]. \end{cases}$$

※ Let $\mathfrak{g}$ be a Lie algebra, V a vector space.

A Lie alg hom $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ is called a representation. We say V is a representation, or just say V is a $\mathfrak{g}$-module.

<u>Example</u> ① $\mathfrak{g}$ is a $\mathfrak{g}$-module via ad-representation.

② For $\mathfrak{g} = \mathfrak{sl}(V)$, we have inclusion $i : \mathfrak{g} \to \mathfrak{gl}(V)$.

Then it is clearly a Lie alg hom.

Hence V is a $\mathfrak{sl}(V)$-module.

※ We denote bilinear $\cdot : \mathfrak{g} \times V \to V$, $x \cdot v = \phi(x)(v)$ for $x \in \mathfrak{g}$, $v \in V$, $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$.

Then ϕ is a representation

$$\iff x\cdot(y\cdot v) - y\cdot(x\cdot v) = [x,y]\cdot v$$

for all $x, y \in \mathfrak{g}$ and $v \in V$.

<u>Example</u> ① $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ induces
$$\begin{aligned} \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ (x, v) &\mapsto [x, v] \end{aligned}$$

② $i : \mathfrak{sl}(V) \to \mathfrak{gl}(V)$ induces
$$\begin{aligned} \mathfrak{sl}(V) \times V &\to V \\ (x, v) &\mapsto x \cdot v \end{aligned}$$

which is just a matrix multiplication.

⊠ ~~For~~ representations V^{ϕ} and W^{ψ}, we define

① dual representation : $\phi^* : \mathfrak{g} \longrightarrow \mathfrak{gl}(V^*)$

$$(x \cdot f)(v) := -f(x \cdot v), \quad \begin{cases} x \in \mathfrak{g} \\ f \in V^* \\ v \in V \end{cases}$$

② direct sum : $\phi \oplus \psi : \mathfrak{g} \to \mathfrak{gl}(V \oplus W)$

$$x \cdot (v + w) := x \cdot v + x \cdot w$$

③ tensor product : $\phi \otimes \psi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V \otimes W)$

$$x \cdot (v \otimes w) := x \cdot v \otimes w + v \otimes x \cdot w \qquad (v \in V,\ w \in W)$$

④ Hom space : $\mathrm{Hom}(\phi, \psi) : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathrm{Hom}(V, W))$

$$(x \cdot f)(v) := x \cdot (f(v)) - f(x \cdot v), \quad \begin{cases} x \in \mathfrak{g} \\ f \in \mathrm{Hom}(V, W) \\ v \in V \end{cases}$$

~~Note~~ $\cancel{V^* \otimes W = \mathrm{Hom}(V, W)}$

$\cancel{f \otimes w \longleftrightarrow (f \otimes w)(v) := f(v)\, w}$ ~~rank 1 operator~~

~~gives a mo~~

⊠ For representations V, W; a linear map $f : V \to W$ is a $\mathfrak{g}$-module map if

$$f(x \cdot v) = x \cdot (f(v)) \quad \text{for all } x \in \mathfrak{g},\ v \in V.$$

Or we call a Lie-homomorphism.

⊠ We have notions of mono-, epi-, isom-.

$\boxed{*}$ $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ a repn.

A <u>subspace</u> W of V is called a <u>submodule</u> if

$$x \cdot w \in W \quad \text{for all } x \in \mathfrak{g},\ w \in W$$

<u>Example</u> $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$

A subspace $\mathfrak{p}$ of $\mathfrak{g}$ is an ad-submodule if

$$[x, w] \in \mathfrak{p} \quad \text{for all } x \in \mathfrak{g},\ w \in \mathfrak{p},$$

that is, $\mathfrak{p}$ is an ideal of $\mathfrak{g}$.

$\boxed{*}$ $f : V \to W$ $\mathfrak{g}$-module map.

① $\mathrm{Ker}\, f$ is a submodule of V,

② $\mathrm{Im}\, f$ is a submodule of W.

$\boxed{*}$ A repn V is called irreducible if

V has no nontrivial submodules (say, $\{0\}$ and V).

<u>Example</u> $\mathfrak{g}$-module $\mathfrak{g}$ is irreducible if and only if

it is a simple Lie algebra.

(If $\mathfrak{g}$ is NOT abelian)

<u>Note</u> $\mathfrak{g}$: abelian $\Leftrightarrow$ ad is a zero map.

Lemma [Schur] $\mathfrak{g}$: Lie alg, V : repn

V : irreducible, $f : V \to V$ $\mathfrak{g}$-module map.

$\Rightarrow$ $f = c \cdot id_V$ for some scalar $c \in \mathbb{C}$.

⟨pf⟩ choose eigen-value $\lambda \in \mathbb{C}$ of f.

$\Rightarrow g = f - c \cdot id_V : V \to V$ also a $\mathfrak{g}$-module map

$Ker(g) \leq V$, $Im(g) \leq V$.

since V is irreducible and $Ker(g) \neq 0$, we must obtain

$Ker\, g = V$, $Im(g) = 0$, i.e. $g = 0$, equivalently, $f = c \cdot id_V$ □

Theorem [Weyl] $\mathfrak{g}$: (f.d.) Lie alg, V : (f.d.) repn

$\Rightarrow$ V is isomorphic to a direct sum of its irreducible submodules.

※ $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ repn. $\Rightarrow$ trace form of ϕ is defined by

$tr_\phi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, $tr_\phi(x, y) = tr(\phi(x) \circ \phi(y))$.

$\Rightarrow$ ① bilinear, symmetric form

② invariant in the sense that $tr_\phi([x, y], z) = tr_\phi(x, [y, z])$

· $tr_{ad}(x, y)$ is called the Killing form κ of $\mathfrak{g}$, that is,

$\kappa(x, y) = tr(ad\, x \circ ad\, y)$

※ $\mathfrak{g} = \mathfrak{sl}(2) = \mathfrak{sl}_2$ has three basis

$$H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad E = \begin{pmatrix} & 1 \\ & \end{pmatrix}, \quad F = \begin{pmatrix} & \\ 1 & \end{pmatrix}.$$

We have presentation

$$\mathfrak{g} = \langle H, E, F \,;\, [H,E] = 2E,\ [H,F] = 2F,\ [E,F] = H \rangle.$$

We fix an order of $\mathfrak{g}$: E, H, F

※ $U(\mathfrak{sl}_2)$ is the associative algebra generated by $1, H, E, F$ subjecting to the relations

$$H \cdot E - E \cdot H = 2E, \quad H \cdot F - F \cdot H = 2F, \quad E \cdot F - F \cdot E = H$$

Note ① $H \cdot E \notin \mathfrak{sl}_2$, ~~in gen~~

② We can define $U(\mathfrak{g})$ for general Lie alg $\mathfrak{g}$.
(or construct using ~~the~~ some universal property)

③ $U(\mathfrak{g})$ -module $\Longleftrightarrow$ $\mathfrak{g}$- module

Thm [Poincare - Birkhoff - Witt]
$U(\mathfrak{sl}_2)$ has basis $\{ E^l H^m F^n \,;\, l, m, n \in \mathbb{Z}_{\geq 0} \}$

$\uparrow$ $\exists$ more general thm.

<u>Thm</u> $U(\mathfrak{sl}_2)$ is a Hopf algebra w/

$$\begin{cases} \Delta(x) = x \otimes 1 + 1 \otimes x, & (x \in \mathfrak{sl}_2) \qquad (\Delta(1) = 1 \otimes 1) \\ \varepsilon(x) = 0, & (x \in \mathfrak{sl}_2) \qquad (\varepsilon(1) = 1) \\ S(x) = -x & \qquad (S(1) = 1) \end{cases}$$

<Pf> It is well-defined, i.e.

$$\Delta(H \cdot E - E \cdot H) \overset{?}{=} \Delta(H)\Delta(E) - \Delta(E)\Delta(H)$$

$$= (H \otimes 1 + 1 \otimes H) \cdot (E \otimes 1 + 1 \otimes E) - (E \otimes 1 + 1 \otimes E)(H \otimes 1 + 1 \otimes H)$$

$$= (HE - EH) \otimes 1 + 1 \otimes (HE - EH)$$

<u>e.g.</u>

$$\begin{array}{ccc} \Delta([x, y]) & \overset{?}{=} & \Delta(x \cdot y - y \cdot x) \\ \| & & \| \\ [x, y] \otimes 1 + 1 \otimes [x, y] & & \Delta(x)\Delta(y) - \Delta(y)\Delta(x) \end{array}$$

① $\begin{cases} (\mathrm{id} \otimes \Delta) \circ \Delta(x) = (\mathrm{id} \otimes \Delta)(x \otimes 1 + 1 \otimes x) = x \otimes \Delta(1) + 1 \otimes \Delta(x) \\ (\Delta \otimes \mathrm{id}) \circ \Delta(x) \end{cases}$

$$= x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x$$

② $\begin{cases} (\varepsilon \otimes \mathrm{id}) \circ \Delta(x) = (\varepsilon \otimes \mathrm{id})(x \otimes 1 + 1 \otimes x) = 0 \otimes 1 + 1 \otimes x \\ x \longmapsto 1 \otimes x \end{cases}$

③ $(\mathrm{id} \otimes \varepsilon) \circ \Delta(x) = x \otimes 1$

④ $m \circ (S \otimes \mathrm{id}) \circ \Delta(x) = m \circ (S \otimes \mathrm{id})(x \otimes 1 + 1 \otimes x) = m \circ (-x \otimes 1 + 1 \otimes x)$

$$= -x \cdot 1 + 1 \cdot x = 0 = (i \circ \varepsilon)(x)$$

⑤ $m \circ (\mathrm{id} \otimes S) \circ \Delta(x) = (i \circ \varepsilon)(x)$

~~For $x=1$, we can check them trivially.~~

Def $U_q(sl_2)$ is an associative alg w/ generators

$1, K^{\pm 1}, E, F$ subjecting to the relations

$$\begin{cases} K^{+1}\cdot K^{-1} = K^{-1}\cdot K^{+1} = 1 \\ K\cdot E = qE\cdot K, \quad K\cdot F = q^{-1}F\cdot K \\ E\cdot F - F\cdot E = \dfrac{K-K^{-1}}{q^{1/2}-q^{-1/2}} \end{cases}$$

* In Lie theory, we usually use $q \rightsquigarrow q^2$ version.

Thm $U_q(sl_2)$ is a Hopf algebra via

$$\begin{cases} \Delta(K^{\pm}) = K^{\pm}\otimes K^{\pm} \\ \Delta(E) = 1\otimes E + E\otimes K \\ \Delta(F) = F\otimes 1 + K^{-1}\otimes F \end{cases} \quad \begin{cases} \varepsilon(K^{\pm}) = 1 \\ \varepsilon(E) = 0 \\ \varepsilon(F) = 0 \end{cases} \quad \begin{cases} S(K^{\pm}) = K^{\mp} \\ S(E) = -EK^{-1} \\ S(F) = -KF \end{cases}$$

Rmk We regard $\begin{cases} q = e^{\hbar} \\ K = q^{H/2} \end{cases}$. (Recall $H\cdot E = E\cdot(H+2)$)

$$K\cdot E = q^{H/2}\cdot E = \sum_n \frac{1}{n!}\left(\frac{\hbar}{2}H\right)^n\cdot E = \sum_n \frac{1}{n!}\left(\frac{\hbar}{2}\right)^n\cdot \underline{H^n\cdot E}$$

$$= \sum_n \frac{1}{n!}\left(\frac{\hbar}{2}\right)^n\cdot E\,(H+2)^n = E\cdot q^{(H+2)/2} = E\cdot K\cdot q^1 = qEK.$$

$*$ Representation Theory of sl_2, i.e. of $U(sl_2)$.

Let $\rho = \rho_V : sl_2 \longrightarrow gl(V)$ be a f.d. Repn.

Think $\rho(H) : V \longrightarrow V$

$v \longmapsto H \cdot v$

$\boxed{\text{Prop}}$ $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$ where $V_\lambda = \{ v \in V ;\ H \cdot v = \lambda \cdot v \}$.

· ~~Choose $v_\lambda \in V_\lambda$ such that~~

$\boxed{\text{Lem}}$ $v \in V_\lambda \Rightarrow E \cdot v \in V_{\lambda+2}, \quad F \cdot v \in V_{\lambda-2}$.

<pf> $H \cdot (E \cdot v) = E \cdot H \cdot v + [H, E] \cdot v$

$= E(\lambda \cdot v) + 2E \cdot v = (\lambda+2) E \cdot v$

$*$ Now chose $v_\lambda \in V_\lambda \setminus \{0\}$ such that $E \cdot v_\lambda = 0$.

Define $w_k = \frac{1}{k!} F^k \cdot v_\lambda$. (i.e. $w_0 = v_\lambda$)

$\boxed{\text{Lemma}}$ ① $H \cdot w_k = (\lambda - 2k) w_k$

② $F \cdot w_k = (k+1) w_{k+1}$

② $E \cdot w_k = (\lambda - k + 1) w_{k-1}$

$\boxed{k>0}$

⟨pf⟩ ① $H\cdot\frac{1}{k!}F^k\cdot v_\lambda = \frac{1}{k!}H\cdot F\cdot(F^{k-1}v_\lambda) = \frac{1}{k} H\cdot F\cdot w_{k-1}$

$= \frac{1}{k}\left(F\cdot H\cdot w_{k-1} + [H,F]\cdot w_{k-1}\right)$

$= \frac{1}{k}\left(F\cdot(\lambda-2(k-1))w_{k-1} - 2F\cdot w_{k-1}\right) = (\lambda-2k\cancel{+2}\cancel{-2})w_k$

② Trivial

③ $\frac{1}{k!}E\cdot F^k v_\lambda = \frac{1}{k}E\cdot F\cdot w_{k-1} = \frac{1}{k}\left(F\cdot E\cdot w_{k-1} + [E,F]\cdot w_{k-1}\right)$

$\boxed{k>0}$ $= \frac{1}{k}\left(F\cdot(\lambda-(k-1)+1)w_{k-2} + H\cdot w_{k-1}\right)$

$= (\lambda-k+2)\cdot\frac{(k-1)}{k!}F^{k-1}v_\lambda + \frac{1}{k}\cdot(\lambda-2(k-1))\,w_{k-1}$

$= \left(\frac{k-1}{k}(\lambda-k+2) + \frac{\lambda-2k+2}{k}\right)w_{k-1}$

$= (\lambda-k+1)\,w_{k-1}$ by induction.

* $w_0,\cdots,w_m^{\neq 0}, \underset{\sim\sim\sim}{w_{m+1}=0} \Rightarrow$
① $\{w_0,\cdots,w_m\}$ is linearly independent
② $\underset{\overset{\shortparallel}{0}}{E\cdot w_{m+1}} = (\lambda-m)w_m \rightsquigarrow \boxed{\lambda=m}$

<u>Def</u> $V_{m+1} = \langle w_0,\cdots,w_m \,;\,$ relations in above lemma $\rangle$

$H \Leftrightarrow \begin{pmatrix} m & & & \\ & m-2 & & \\ & & \ddots & \\ & & & -m \end{pmatrix}$, $F \Leftrightarrow \begin{pmatrix} 0 & & & & \\ 1 & \ddots & & & \\ & 2 & \ddots & & \\ & & \ddots & \ddots & \\ & & & m & 0 \end{pmatrix}$ $E \Leftrightarrow \begin{pmatrix} 0 & m & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}$

* $\dim(V_{m+1}) = m+1$

<u>Thm</u> $V(m) = V_{m+1}$ exhausts the irred repns of $U(sl_2)$.

<u>Thm</u> The following correspondence gives f.d. $U_q(sl_2)$-repns $V_q(m) = V^q_{m+1}$

$$K \Leftrightarrow \begin{pmatrix} q^{m/2} & & & \\ & q^{m/2-1} & & \\ & & \ddots & \\ & & & q^{-m/2} \end{pmatrix} \qquad F \Leftrightarrow \begin{pmatrix} 0 & & & & \\ [1] & 0 & & & \\ & [2] & \ddots & & \\ & & \ddots & \ddots & \\ & & & [m] & 0 \end{pmatrix}$$

$$E \Leftrightarrow \begin{pmatrix} 0 & [m] & & \\ & 0 & \ddots & \\ & & \ddots & [1] \\ & & & 0 \end{pmatrix}$$

where $[m] = \dfrac{q^{m/2} - q^{-m/2}}{q^{1/2} - q^{-1/2}}$, the modified quantum integers.

<u>e.g.</u>

$$K \cdot E = \begin{pmatrix} q^{\frac{m}{2}} & & & \\ & q^{\frac{m-2}{2}} & & \\ & & \ddots & \\ & & & q^{-\frac{m}{2}} \end{pmatrix} \begin{pmatrix} 0 & [m] & & \\ & 0 & \ddots & \\ & & \ddots & [1] \\ & & & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & q^{\frac{m}{2}}[m] & & & \\ & 0 & q^{\frac{m-2}{2}}[m-1] & & \\ & & \ddots & \ddots & \\ & & & \ddots & q^{-\frac{m-2}{2}}[1] \\ & & & & 0 \end{pmatrix}$$

$$\overline{q \cdot E \cdot K} = \cancel{q} \begin{pmatrix} 0 & [m] & & \\ & \ddots & \ddots & \\ & & \ddots & [1] \\ & & & 0 \end{pmatrix} \begin{pmatrix} q^{\frac{m+2}{2}} & & & \\ & q^{\frac{m}{2}} & & \\ & & \ddots & \\ & & & q^{-\frac{m-2}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & [m]q^{\frac{m}{2}} & & \\ & \ddots & \ddots & \\ & & \ddots & [1] \cdot q^{-\frac{m-2}{2}} \\ & & & 0 \end{pmatrix}$$