Volumes of polyhedra and cone-manifolds in spaces of constant curvature

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1 Hyperbolic, spherical and Euclidean spaces

According to Y. Wolf [1], a simply-connected Riemannian manifold X is of constant curvature if and only if its isometry group acts transitively on X and the isometry group coinsides with the group of all orthogonal transformation of the tangent space.

A manifold X is hyperbolic, spherical or Euclidean if it has negative, positive or zero curvature, respectively. By Hadamard theorem, a complete simply-connected manifold X of a given curvature κ is unique up to isometry. We denote the respective manifolds with curvature $\kappa = -1, 0, +1$ by $\mathbb{H}^n, \mathbb{E}^n$, and $\mathbb{S}^n$. Here, $n = \dim X$

We note that $\mathbb{H}^n$ is also known as Lobachevski, Bolyai, and Gaussian space.

Following to Y.W. Cannon, W.Y. Floyd, R.Kenyon, and W.R. Parry [2], we introduce the following five models for hyperbolic space:

- **H** the half-space model
- **I** the interior of the disk model
- **Y** the Yemisphere model (**Y** is as in Spanish)
- **K** the Klein model
- **L** the 'Loid model (short for hyperboloid)

Each model is defined on a different subsets of $\mathbb{R}^{n+1}$, called its domain; for $n = 1$, these sets are schematically indicated in Fig. 1, which can also be regarded as a cross section of the picture in higher dimensions.

Here are the definitions of the five domains:

$$\begin{aligned}
\mathbf{H} &= \{(1, x_2, \ldots, x_{n+1}) : x_{n+1} > 0\} \\
\mathbf{I} &= \{(x_1, \ldots, x_n, 0) : x_1^2 + \ldots + x_n^2 < 1\} \\
\mathbf{Y} &= \{(x_1, \ldots, x_{n+1}) : x_1^2 + \ldots + x_{n+1}^2 = 1,\ x_{n+1} > 0\} \\
\mathbf{K} &= \{(x_1, \ldots, x_n, 1) : x_1^2 + \ldots + x_n^2 < 1\} \\
\mathbf{L} &= \{(x_1, \ldots, x_n, x_{n+1}) : x_1^2 + \ldots + x_n^2 - x_{n+1}^2 = -1\}.
\end{aligned}$$

The associated Riemannian metrics ds^2 that complete the analytic description of the five models are :

$$\begin{aligned}
ds^2_{\mathbf{H}} &= \frac{dx_2^2 + \ldots + dx_{n+1}^2}{x_{n+1}^2} \\
ds^2_{\mathbf{I}} &= 4\frac{dx_1^2 + \ldots + dx_n^2}{(1 - x_1^2 - \ldots - x_n^2)^2} \\
ds^2_{\mathbf{Y}} &= \frac{dx_1^2 + \ldots + dx_{n+1}^2}{x_{n+1}^2} \\
ds^2_{\mathbf{K}} &= \frac{dx_1^2 + \ldots + dx_{n+1}^2}{1 - x_1^2 - \ldots - x_n^2} + \frac{(x_1 dx_1 + \ldots + x_n dx_n)^2}{(1 - x_1^2 - \ldots - x_n^2)^2} \\
ds^2_{\mathbf{L}} &= dx_1^2 + \ldots + dx_n^2 - dx_{n+1}^2.
\end{aligned}$$

To see that these five models are isometrically equivalent, we need to describe isometries among them. We use $\mathbf{Y}$ as the cetral and describe for each of the others as simple map to or from $\mathbf{Y}$.

The map $\alpha : \mathbf{Y} \longrightarrow \mathbf{H}$ is central projection from the point $(-1, 0, \ldots, 0)$:

$$\alpha \ : \ \mathbf{Y} \longrightarrow \mathbf{H}, \ (x_1, \ldots, x_{n+1}) \mapsto \left(1, \frac{2x_2}{x_1 + 1}, \ldots, \frac{2x_{n+1}}{x_1 + 1}\right)$$

The map $\beta : \mathbf{Y} \longrightarrow \mathbf{I}$ is central projection from the point $(0, \ldots, 0, -1)$:

$$\beta \ : \ \mathbf{Y} \longrightarrow \mathbf{I}, \ (x_1, \ldots, x_{n+1}) \mapsto \left(1, \frac{x_1}{x_{n+1} + 1}, \ldots, \frac{x_n}{x_{n+1} + 1}, 0\right)$$

The map $\gamma : \mathbf{K} \longrightarrow \mathbf{Y}$ is vertical projection:

$$\gamma \ : \ \mathbf{K} \longrightarrow \mathbf{Y}, \ (x_1, \ldots, x_n, 1) \mapsto \left(x_1, \ldots, x_n, \sqrt{1 - x_1^2 - \ldots - x_n^2}\right)$$

The map $\delta : \mathbf{L} \longrightarrow \mathbf{Y}$ is central projection from the point $(0, \ldots, 0, -1)$:

$$\delta \ : \ \mathbf{L} \longrightarrow \mathbf{Y}, \ (x_1, \ldots, x_{n+1}) \mapsto \left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}, \frac{1}{x_{n+1}}\right)$$

The standard model for $\mathbb{S}^n$ comes from its natural embedding into $\mathbb{R}^{n+1}$ as a set

$$\{(x_1, \ldots, x_{n+1}) \ : \ x_1^2 + \ldots + x_n + 1^2 = 1\}$$

with induced metric

$$ds^2_{\mathbb{S}^n} = dx_1^2 + \ldots + dx_{n+1}^2.$$

The Euclidean space $\mathbb{E}^n$ can be realized as a set $\mathbb{R}^n$ with metric $ds^2 = dx_1^2 + \ldots + dx_n^2$.

2 Schläfli variation formula

2.1 Schläfli formula for polyhedra

In 1866 Schläfli discovered a remarkable formula for the volume of n-dimensional spherical cone-manifold. It was done before the notion of manifold became known and wide understandable. Later, H. Kneser[1936] recognized that the same formulas remain to be true also for hyperbolic geometry. Essential influence for finding of correct relationship between Lobachevsky and Schläfli results on volumes of hyperbolic and spherical tetrahedras are done by H.S.M. Coxeter[1935].

We represent the result of Schläfli in the form given by Y. Milnor in [3].

Theorem 2.1 (Schläfli variation formula for polyhedra) *Let X^n be a space of constant curvature κ. Consider a family of (convex) polyhedra P depending on one or more parameters in a differential manner and keeping the same topological type. Then the derivative of volume of P satisfies*

$$(n-1)\kappa dP = \sum_F V_{n-1}(F) d\theta(F)$$

where the sum is taken over all $(n-1)$-faces (facets) of P. $V_{n-1}(F)$ is $(n-1)$-dimensional volume of F, and $\theta(F)$ is the interior angle along F.

We note that condition for polyhedra P to be convex is not necessary, but correct definition of nonconvex polyhedron is very delicated problem. See books and papers by F. Grünbaum on this subject. Everytime P is supposed to be compact polyhedron with finite number of faces.

2.2 Schläfli formula for cone-manifolds

Cone-manifold is a metric space locally isometric to polyhedron with partially identified faces. The geometry of cone-manifold is defined by geometry of polyhedron.

Example

1. The Poincare homology sphere
   Can be obtained from a regular spherical $\frac{2\pi}{3}$-dodecahedron by identification of its opposite faces.

2. Hyperbolic Seifert-Weber Space
   Can be obtained from a regular hyperbolic $\frac{2\pi}{5}$-dodecahedron by identification of its opposite faces.

3. Borromean Ring orbifold
   Euclidean geometry

4. Coxeter polyhedron and Coxeter orbifold are uniquely defined by Coxeter scheme.

5. All three geometries appear. [4] (On some generalized triangular groups and three-dimensional orbifolds)

6. Pleated surface [5]
Genus three surface is obtained as a union of 12 hexagonal faces of two regular truncated tetrahedra.

The following theorem was proved by C.D. Hodgson [6].

Theorem 2.2 (Schläfli variation formula for cone-manifolds) *Suppose that C_t is a smooth 1-parameter family of (curvature κ) cone-manifold structures on a n-manifold with locus Σ of a fixed topological type. Then the derivative of volume of C_t satisfies*

$$(n-1)\kappa dV(C_t) = \sum_{\sigma} V_{n-2}(\sigma) d\theta(\sigma)$$

where the sum is taken over all components σ of the singular locus Σ and $\theta(\sigma)$ is the cone angle along σ.

Remark 1 *To be sure that $V_{n-2}(\sigma) < \infty$, all $(n-2)$-dimensional component σ of Σ are supposed to be compact.*

2.3 Schläfli formula for cusped cone-manifolds

The following very convenient for application version of variation formula was suggested by I. Rivin and C. Hodgson

Theorem 2.3 (Schläfli variation formula for cusped cone-manifolds) *Let C be a cusped hyperbolic cone-manifold of finite volume. Suppose that C has cusps $1, 2, \ldots, K$ and $O_1, O_2, \ldots, O_K$ is a set of non-overlapping horoballs in these cusps, respectively. Then the volume $vol(C)$ is given by the Schläfli formula applied to truncated cone-manifold $\widetilde{C} = C \setminus \bigcup_{j=1}^{K} O_j$. The result does not depend of particular choice of horoballs $O_1, O_2, \ldots, O_K$.*

Sketch of proof. First of all, we show that the result of applying Schläfli formula is independent from a choice of horoball. We realize a horoball neighborhood O_1 as a subset of the upper half space model of $\mathbb{H}^3$ with a cusp on infinity.

Let $l_1, l_2, \ldots, l_S$ be the lengths of edges of the singular locus of $C \setminus O_1$ terminated at the cusp on infinity and $\alpha_1, \alpha_2, \ldots, \alpha_S$ are respective cone-angles. Then $\partial O_1 \cap C$ is a Euclidean S-gon (or two copies of it if $\partial O_1 \cap S$ is two-dimensional sphere.) We have $\alpha_1 + \alpha_2 + \ldots + \alpha_S = (S-2)\pi$ (or $\alpha_1 + \alpha_2 + \ldots + \alpha_S = 2(S-2)\pi$ in the second case.) In both cases

$$d\alpha_1 + d\alpha_2 + \ldots + d\alpha_S = 0$$

during small deformations of cone-manifold C.

The input of cups 1 in the Schläfli formula is given by $\sum_{i=1}^{S} l_i \, d\alpha_i$ if O_1 is deleted, and $\sum_{i=1}^{S}(l_i + \Delta l)d\alpha_i$ if $\widetilde{O}_1$ is deleted. Since

$$\sum_{i=1}^{S} \Delta l \, d\alpha_i = \Delta l \cdot d\left(\sum_{i=1}^{S} \alpha_i\right)$$

the input does not depend on the choice of horoball O_1.

Let $\widetilde{O}_1$, $\widetilde{O}_1 \subset O_1$ be another horoball at the cusp on infinity. Then horosphere $\partial\widetilde{O}_1$ and ∂O_1 are equidistance surfaces. We assume that $\partial\widetilde{O}_1$ is obtained from ∂O_1 by a vertical shift Δl, $\Delta l > 0$. See Fig.

Let $\widetilde{O}_1 = \widetilde{O}_1(n)$ be obtained from O_1 by vertical shift $\Delta l = n$ and $\widetilde{C}_n = C \backslash \widetilde{O}_1(n)$. Then $\bigcup_n \widetilde{C}_n = C$, $Vol(C) < \infty$, and by standard properties of measure we have

$$Vol(\widetilde{C}_n) \longrightarrow Vol(C) \; , \; n \longrightarrow \infty$$

Now we note that $\widetilde{C}$ is truncated but not geodesic cone-manifold. By a slight modification of truncated boundary of $\widetilde{C}_n$ we obtain a closed geodesic cone-manifold G_n with $Vol(G_n) = Vol(\widetilde{C}_n) + O_n$ where $O_n \to 0$ as $n \to \infty$.

The volume of G_n is given by the previous version of variation theorem and we are done.

□

2.4 The volume of ideal tetrahedron

We illustrate the previous theorem by a short prove of Milnor formula for the volume of ideal tetrahedron.

Theorem 2.4 (Milnor, 1982) *The volume of an ideal hyperbolic tetrahedron $T = T(A, B, C)$ with angles A, B, C $(A + B + C = \pi)$ is given by the formula*

$$Vol(T) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma)$$

where $\Lambda(x) = -\int_0^x \log|2\sin\xi| d\xi$ is the Lobachevsky function.

Proof. We realize the tetrahedron T is the upper space model with one vertex on the infinity. See Fig. Then the projection of tetrahedron on the plane is an Euclidean triangle ABC with angles α, β, γ and edges a, b, c. We suppose that $\alpha \le \beta \le \gamma$, then also $a \le b \le c$. Delete horoballs of diameter a at the cusps A, B, and C.

Their projection on the plane are shown on Fig. Also we cut the cusp at the infinity by horosphere of the height h, $h > a$.

For any h, the input of vertical edges is Schläfli formula coincides with those for $h = a$ and is equal to zero.

The edge BC of tetrahedron T after removing of horoball collapses to a point and also has no influence to Schläfli formula. Only two edges l_β and l_γ are remained. To find these lengths we need the following elementary lemma.

Lemma 2.5 *Let $0 \leq \tau \leq R$ and $z_1,\ z_2$ are two points on the upper half plane chosen as shown on Fig.*

Then the hyperbolic distance between z_1 and z_2 is

$$\rho(z_1, z_2) = 2\log\frac{R}{r}.$$

Proof. We note that $z_1 = (-x, y)$ and $z_2 = (x, y)$, where x and y satisfy

$$(x-R)^2 + (y-r)^2 = r^2 \text{ and } x^2 + y^2 = R^2.$$

Solving the system of equations we have

$$x = R\frac{R^2 - r^2}{R^2 + r^2} \text{ and } y = \frac{2R^2 r}{R^2 + r^2}.$$

Then, by A. Beardon([]) we obtain

$$\cosh\rho(z_1, z_2) = 1 + \frac{|z_1 - z_2|^2}{2\mathrm{Im}z_1\mathrm{Im}z_2} = 1 + \frac{(R^2 - r^2)^2}{2R^2r^2} = \frac{1}{2}\left(\left(\frac{R}{r}\right)^2 + \left(\frac{r}{R}\right)^2\right).$$

Hence,

$$\rho(z_1, z_2) = \log\left(\frac{R}{r}\right)^2 = 2\log\frac{R}{r}.$$

□

Applying lemma to calculate the length l_β we have

$$R = \frac{b}{2},\ r = \frac{a}{2}.$$

Hence,

$$l_\beta = 2\log\frac{b}{a} = 2\log\frac{\sin\beta}{\sin\alpha}.$$

Similarly,

$$l_\gamma = 2\log\frac{c}{a} = 2\log\frac{\sin\gamma}{\sin\alpha}.$$

Let $V = Vol(T(\alpha, \beta, \gamma))$. We note that $\alpha + \beta + \gamma = \pi$. By Schläfli formula we have;

$$\begin{aligned} -dV &= \frac{1}{2}l_\beta d\beta + \frac{1}{2}l_\gamma d\gamma = \log\frac{2\sin\beta}{2\sin\alpha}d\beta + \log\frac{2\sin\gamma}{2\sin\alpha}d\gamma \\ &= \log(2\sin\beta)d\beta + \log(2\sin\gamma)d\gamma - \log(2\sin\alpha)(d\beta + d\gamma) \\ &= \log(2\sin\alpha)d\alpha + \log(2\sin\beta)d\beta + \log(2\sin\gamma)d\gamma. \end{aligned}$$

After integration we have well-known Milnor formula

$$V = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma)$$

where $\Lambda(x) = -\int_0^x \log|2\sin\xi| d\xi$.

$\square$

Exercise 1 *Applying the above arguments, show that the volume of an ideal piramid* $P = P(\alpha_1, \ldots, \alpha_n)$, $n \geq 3$ *with bottom dihedral angles* $\alpha_1, \ldots, \alpha_n$ *satisfying* $\alpha_1 + \ldots + \alpha_n = \pi$ *(See fig) is given by the formula [Thurston, Vinberg]*

$$Vol(P) = \Lambda(\alpha_1) + \ldots + \Lambda(\alpha_n)$$

2.5 Ideal symmetric octahedron

Let O be an ideal symmetric octahedron with all vertices on the infinity.

Then $C = \pi - A$, $D = \pi - B$, $F = \pi - E$ and hyperbolic volume of O is given by the following:

Theorem 2.6 (Yana Mohanty, 2002)

$$\begin{aligned} Vol(O) &= 2\left(\Lambda\left(\frac{\pi + A + B + E}{2}\right) + \Lambda\left(\frac{\pi - A - B + E}{2}\right)\right. \\ &\quad \left. + \Lambda\left(\frac{\pi + A - B - E}{2}\right) + \Lambda\left(\frac{\pi - A + B - E}{2}\right)\right) \end{aligned}$$

Proof. Joint top and bottom vertices of O by infinite geodesic line. Then O is divided into four ideal tetrahedra sharing a common edge. See a projection on Fig.

The common edge is shown by point $\mathcal{O}$, while the edges with dihedral angles A, B, C, D by points $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ respectively. Since the opposite angle of ideal tetrahedron are equal, we know all dihedral angle E, F, E, F along edge $\mathcal{O}$.

Let we introduce notation by remained dihedral angles as shown on Figure.

By the sine rule we have

$$\frac{\sin x}{\sin t'} = \frac{\mathcal{OD}}{\mathcal{OA}}, \quad \frac{\sin y}{\sin x'} = \frac{\mathcal{OA}}{\mathcal{OB}}, \quad \frac{\sin z}{\sin y'} = \frac{\mathcal{OB}}{\mathcal{OC}}, \quad \frac{\sin t}{\sin z'} = \frac{\mathcal{OC}}{\mathcal{OD}}$$

Multiplying the equation we obtain

$$\frac{\sin x}{\sin t'} \cdot \frac{\sin y}{\sin x'} \cdot \frac{\sin z}{\sin y'} \cdot \frac{\sin t}{\sin z'} = 1 \tag{1}$$

From each of four triangles we have

$$x + t' + F = \pi, \; y + x' + E = \pi, \; z + y' + F = \pi, \; t + z' + E = \pi, \text{ and}$$

$$x + x' = A, \; y + y' = B, \; z + z' = B, \; z + z' = C.$$

We suppose that x is given and find all other unknown angles through x and angles of O. We have

$$x = x,\ x' = A - x,\ y = \pi + x - A - E,\ y' = -x + A + B + E$$

$$z = \pi + x - A - B,\ z' = -x + B,\ t = \pi + x - B - E,\ t' = -x + E$$

From the main equation (1) for $x = -u$, we obtain

$$\frac{\sin u \sin(A + B + u) \sin(A + E + u) \sin(B + E + u)}{\sin(A + u) \sin(B + u) \sin(E + u) \sin(A + B + E + u)} = 1$$

The equation is equivalent

$$\frac{(\cos(A + B) - \cos(A + B + 2u))(\cos(A - B) - \cos(A + B + 2E + 2u))}{(\cos(A - B) - \cos(A + B + 2u))(\cos(A + B) - \cos(A + B + 2E + 2u))} = 1$$

which gives

$$\cos(A + B + 2u) - \cos(A + B + 2E + 2u) = 0$$

simply

$$\sin(A + B + E + 2u) = 0.$$

Returning to x, we have

$$\sin(A + B + E - 2x) = 0.$$

Hence

$$2x = A + B + E + \pi k$$

for some integer k. Note that for $A = B = E = \frac{\pi}{2}$ we have $x = \frac{\pi}{4}$. Hence $k = -1$ and

$$x = \frac{A + B + E - \pi}{2}.$$

Symmetric octahedron is the union of two congruent tetrahedra with angles

$$x = \frac{A + B + E - \pi}{2},\ t' = \frac{-A - B + E - \pi}{2},\ f = \pi - E,$$

and two congruent tetrahedra with angles

$$x' = \frac{A - B - E + \pi}{2},\ y = \frac{-A + B - E + \pi}{2} \text{ and } E.$$

The result follows from the Milnor formula.

□

2.6 General ideal octahedron

Let O be a general ideal octahedron with all vertices on the infinity. A general ideal octahedron (Fig.) satisfies the following conditions;

$$A' + A + E + H = 2\pi,\ B' + B + E + F = 2\pi,\ C' + C + F + G = 2\pi,$$
$$D' + D + G + H = 2\pi,\ A + B + C + D = 2\pi,\ A' + B' + C' + D' = 2\pi.$$

Excluding A', B', C', D' we have the following relations

$$A + B + C + D = 2\pi,\ E + F + G + H = 2\pi.$$

Hence, O depends on 6 real parameters, say A, B, C, E, F, G.

For two remained, we obtain

$$D = 2\pi - A - B - C,\ H = 2\pi - E - F - G.$$

Consider tesselation of O into four ideal tetrahedra. (Fig.)

Similary to the case of symnmetric octahedron, we have

$$\frac{\sin x \sin y \sin z \sin t}{\sin x' \sin y' \sin z' \sin t'} = 1,$$

where

$$x = x,\ x' = A - x,\ y = \pi + x - A - E,\ y' = -x + A + B + E - \pi,$$
$$z = 2\pi + x - A - B - E - F,\ z' = -x + A + B + C + E + F - 2\pi,$$
$$t = 3\pi + x - A - B - C - E - F - G,\ t' = -x - H + \pi. \quad (2)$$

Putting $x = -u$, we obtain

$$\frac{\sin u \sin(A + E + u) \sin(A + B + E + F + u) \sin(A + B + C + E + F + G + u)}{\sin(A + u) \sin(A + B + E + u) \sin(A + B + C + E + F + u) \sin(E + F + G + u)} = 1 \quad (3)$$

An explicit solution of the above equation is rather complicated. However, the following observation is helpful.

Lemma 2.7 (Derevnin-Mednykh, 2003) *The equation*

$$\frac{\sin(a + u) \sin(b + u) \sin(c + u) \sin(d + u)}{\sin(a' + u) \sin(b' + u) \sin(c' + u) \sin(d' + u)} = 1$$

where $a + b + c + d = a' + b' + c' + d'$ *is quadratic with respect to* $\tan u$.

In our case,

$$a + b + c + d = 3A + 2B + 3E + 2F + G = a' + b' + c' + d'.$$

We summerize the results in the following:

Proposition 2.8 *The volume of a general ideal octahedron O is given by the formula*

$$\begin{aligned} Vol(O) &= \Lambda(E) + \Lambda(F) + \Lambda(G) + \Lambda(H) + \Lambda(x) + \Lambda(x') \\ &+ \Lambda(y) + \Lambda(y') + \Lambda(z) + \Lambda(z') + \Lambda(t) + \Lambda(t') \end{aligned}$$

where $x = -u$, $0 < x < \pi$ *is given by (3) and other variables are defined by (2).*

3 Volumes of compact polyhedra

3.1 Lambert cube

One of the simplest compact polyhedra in the hyperbolic space $\mathbb{H}^3$ is the Lambert cube $Q(\alpha, \beta, \gamma)$. See Fig. By Andreev's theorem, it is hyperbolic for all $0 < \alpha, \beta, \gamma < \frac{\pi}{2}$.

The name of the cube comes from the well-known Lambert quadrilateral which, in turn, is a half of Sacceri- quadrilateral.

Both have been used to disprove the fifth Postulate of Euclid.

Before to deal with the cube, we describe basic analytic ideas on an example of Lambert quadrilateral $Q(\alpha)$.

Let $A, B > 0$. Consider two Lorenzian metrics

$$ds^2 = dx^2 + dy^2 - dt^2 \text{ and } d\sigma^2 = \frac{dx^2}{A^2} + \frac{dy^2}{B^2} - dt^2.$$

Then the map

$$(x, y, t) \mapsto (Ax, By, t)$$

is an isometry of the Lorenzian spaces

$$(\mathbb{R}^{2,1}, ds^2) \mapsto (\mathbb{R}^{2,1}, d\sigma^2).$$

Consider a realization of $Q(\alpha)$ in the space $(\mathbb{R}^{2,1}, d\sigma^2)$ for $t = 1$. (See fig.)

The lines l and m are given in $\mathbb{R}^3$ by the following projective equations

$$\begin{aligned} l &= \{(x, y, t) \ : \ (1-a)x + y - t = 0\}, \\ m &= \{(x, y, t) \ : \ x - t = 0\}. \end{aligned}$$

We introduce in $\mathbb{R}^{2,1}$ the Lorenzian inner product

$$\langle (x, y, t), (x', y', t') \rangle = \frac{xx'}{A^2} + \frac{yy'}{B^2} - tt'.$$

The normal vectors of l and m are defined as

$$N_l = ((1-a)A^2, B^2, 1) \text{ and } N_m = (A^2, 0, 1).$$

Since $l \perp m$, we have

$$\langle N_l, N_m \rangle = (1-a)A^2 - 1 = 0.$$

Hence $1 - a = \frac{1}{A^2}$ and $a = 1 - \frac{1}{A^2}$.

Let $l^\perp$ be a line symmetric to l with respect to y-axis. Then

$$N_l = (1, B^2, 1) \text{ and } N_{l^\perp} = (-1, B^2, 1).$$

By the definition of $Q(\alpha)$, we have

$$\cos 2\alpha = -\frac{\langle N_l, N_{l^\perp} \rangle}{\langle N_l, N_l \rangle^{\frac{1}{2}} \langle N_{l^\perp}, N_{l^\perp} \rangle^{\frac{1}{2}}} = \frac{\frac{1}{A^2} - B^2 + 1}{\frac{1}{A^2} + B^2 - 1}.$$

Hence

$$\tan^2\alpha = \frac{1-\cos 2\alpha}{1+\cos 2\alpha} = A^2(B^2-1).$$

Denote by δ_a and δ_b the sides of $Q(\alpha)$ lying on x- and y-axis and by δ_c and δ_d the sides opposite to δ_a and δ_b, respectively.

For any $v=(x,y,t)\in\mathbb{R}^{2,1}$, denote by $\|v\|=\langle v,v\rangle^{\frac{1}{2}}$ the Lorenzian norm of v. We note that if $v\in\mathbb{H}^3$, then $\langle v,v\rangle<0$ and the norm $\|v\|=i\,w,\ w>0$ is a positive pure imaginary number.

Since δ_a is the hyperbolic distance between points (0,0,1) and (0,1,1), we have

$$\cosh\delta_a = \frac{\langle(0,0,1),(1,0,1)\rangle}{\|(0,0,1)\|\cdot\|(1,0,1)\|} = \frac{A}{\sqrt{A^2-1}}.$$

In a similar way,

$$\cosh\delta_b = \frac{B}{\sqrt{B^2-1}}.$$

Hence $A=\coth\delta_a$ and $B=\coth\delta_b$. We also have

$$a=\frac{A^2-1}{A^2}=\frac{1}{\cosh^2\delta_a}.$$

For the list of trigonometric identities for $Q(\alpha)$, see books of A. Beardon and E. Vinberg.

In a similar way, we realize a Lambert cube $Q(\alpha,\beta,\gamma)$ as a subset of hyperplane $t=1$ in the Lorenzian space $(\mathbb{R}^{3,1},d\sigma^2)$ with metric

$$ds^2=\frac{dx^2}{A^2}+\frac{dy^2}{B^2}+\frac{dz^2}{C^2}-dt^2,\ \text{where } A,B,C>0.$$

See Fig. and Hilden, Lozano and Montesinos' paper in Topology'90 for detail.

The planes $\mathbf{a},\mathbf{b},\mathbf{c}$ are given by the following projective equations

$$\begin{aligned}\mathbf{a} &= \{(x,y,z,t)\ :\ x+(1-c)z-t=0\},\\ \mathbf{b} &= \{(x,y,z,t)\ :\ (1-a)x+y-t=0\},\\ \mathbf{c} &= \{(x,y,z,t)\ :\ (1-b)y+z-t=0\}.\end{aligned}$$

The respective normal vectors are

$$\begin{aligned}N_{\mathbf{a}} &= (A^2,0,(1-c)C^2,1),\\ N_{\mathbf{b}} &= ((1-a)A^2,B^2,0,1),\\ N_{\mathbf{c}} &= (0,(1-b)B^2,C^2,1).\end{aligned}$$

Since $N_{\mathbf{a}}$, $N_{\mathbf{b}}$, and $N_{\mathbf{c}}$ are mutually orthogonal, we obtain

$$a=1-\frac{1}{A^2},\ b=1-\frac{1}{B^2},\ c=1-\frac{1}{C^2}$$

We endow $\mathbb{R}^{3,1}$ with a Lorenzian inner product

$$\langle (x,y,z,t),(x',y',z',t')\rangle = \frac{xx'}{A^2} + \frac{yy'}{B^2} + \frac{zz'}{C^2} - tt'.$$

By making use of direct calculations, we obtain the following sine-cosine theorem.

Theorem 3.1 (Derevnin, Mednykh, 2000) *Let $Q(\alpha,\beta,\gamma)$ be a hyperbolic Lambert cube with essential angles α,β,γ and edges $l_\alpha, l_\beta, l_\gamma$. Then*

$$\begin{aligned}
\frac{\sin\alpha}{\sinh l_\alpha}\cdot\frac{\sin\beta}{\sinh l_\beta}\cdot\frac{\cos\gamma}{\cosh l_\gamma} &= 1,\\
\frac{\sin\alpha}{\sinh l_\alpha}\cdot\frac{\cos\beta}{\cosh l_\beta}\cdot\frac{\sin\gamma}{\sinh l_\gamma} &= 1,\\
\frac{\cos\alpha}{\cosh l_\alpha}\cdot\frac{\sin\beta}{\sinh l_\beta}\cdot\frac{\sin\gamma}{\sinh l_\gamma} &= 1.
\end{aligned}$$

As a consequence, we obtain the following tangent rule established by A. Mednykh []. Partially this result was contained in the paper by R. Kellerhals (1989).

Theorem 3.2 (Mednykh, 2003) *Let $Q(\alpha,\beta,\gamma)$ be a hyperbolic Lambert cube with essential angles α,β,γ and edges $l_\alpha, l_\beta, l_\gamma$. Then*

$$\frac{\tan\alpha}{\tanh l_\alpha} = \frac{\tan\beta}{\tanh l_\beta} = \frac{\tan\gamma}{\tanh l_\gamma} = T,$$

where T is a positive root of equation

$$T^4 - (A^2+B^2+C^2+1)T^2 - A^2B^2C^2 = 0,$$

with

$$A = \tan\alpha,\ \ B = \tan\beta,\ \ \text{and } C = \tan\gamma.$$

Proof. Dividing the first equation in sine-cosine theorem by the second, we obtain

$$\frac{\tan\beta}{\tanh l_\beta} = \frac{\tan\gamma}{\tanh l_\gamma}.$$

In the same time, dividing the second equation by the third one, we have

$$\frac{\tan\alpha}{\tanh l_\alpha} = \frac{\tan\beta}{\tanh l_\beta}.$$

Hence,

$$\frac{\tan\alpha}{\tanh l_\alpha} = \frac{\tan\beta}{\tanh l_\beta} = \frac{\tan\gamma}{\tanh l_\gamma} = T$$

for some unknown T.

Putting $A = \tan\alpha,\ B = \tan\beta,$ and $C = \tan\gamma$, we obtain

$$\tanh l_\alpha = \frac{A}{T},\ \tanh l_\beta = \frac{B}{T},\ \tanh l_\gamma = \frac{C}{T}.$$

Squaring the first equation in sine-cosine theorem and using elementary trigonometric identities, we get

$$\frac{T^2 - A^2}{1 + A^2} \cdot \frac{T^2 - B^2}{1 + B^2} \cdot \frac{T^2 - C^2}{1 + C^2} \cdot \frac{1}{T^2} = 1.$$

(Compare with the equation in HLM, Topology '90. The later equation is equivalent to

$$(T^2 + 1)(T^4 - (A^2 + B^2 + C^2 + 1)T^2 - A^2B^2C^2) = 0.)$$

Since T is positive, we are done. □

3.2 Volume of Lambert cube

The volume of Lambert cube is given by the following theorem :

Theorem 3.3 (Derevnin, Mednykh, 2002) *Let $Q(\alpha, \beta, \gamma)$ be a hyperbolic Lambert cube with essential angles α, β, and γ. Then the volume of $Q(\alpha, \beta, \gamma)$ is given by the formula*

$$Vol(Q(\alpha, \beta, \gamma)) = \frac{1}{4}\int_T^\infty \log\left(\frac{t^2 - A^2}{1 + A^2} \cdot \frac{t^2 - B^2}{1 + B^2} \cdot \frac{t^2 - C^2}{1 + C^2} \cdot \frac{1}{t^2}\right)\frac{dt}{1 + t^2},$$

where $A = \tan\alpha,\ B = \tan\beta,\ C = \tan\gamma,$ and T is a positive root of the equation

$$T^4 - (A^2 + B^2 + C^2 + 1)T^2 - A^2B^2C^2 = 0.$$

Proof. We note that each face of $Q(\alpha, \beta, \gamma)$ is a Lambert quadrilateral. If $\alpha = \beta = \gamma \to \frac{\pi}{2}$, then face as well as cube itself are collapsing to a point. Hence the volume $V(\alpha, \beta, \gamma) = Vol(Q(\alpha, \beta, \gamma))$ satisfies the following initial condition

$$V(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}) = 0.$$

By the Schläfli formula, we have

$$-dV = \frac{1}{2}(l_\alpha d\alpha + l_\beta d\beta + l_\gamma d\gamma),$$

where $l_\alpha,\ l_\beta,$ and l_γ are the lengths of respective essential edges. One can define a volume as a (unique) solution of the following system of differential equations :

$$\begin{cases} \frac{\partial V}{d\alpha} = -\frac{l_\alpha}{2},\ \frac{\partial V}{d\beta} = -\frac{l_\beta}{2},\ \frac{\partial V}{d\gamma} = -\frac{l_\gamma}{2} \\ V(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}) = 0 \end{cases}$$

Let

$$W = \frac{1}{4}\int_T^{\infty} \log\ \left(\frac{t^2 - A^2}{1 + A^2} \cdot \frac{t^2 - B^2}{1 + B^2} \cdot \frac{t^2 - C^2}{1 + C^2} \cdot \frac{1}{t^2}\right) \frac{dt}{1 + t^2}.$$

We will show that W satisfies the above system of differential equations. Then $W = V$.

We have

$$\frac{\partial W}{\partial \alpha} = \frac{\partial W}{\partial A}\frac{\partial A}{\partial \alpha} = -\frac{1}{2}\text{arctanh}\frac{A}{T} = -\frac{l_\alpha}{2}.$$

The last equation follows from the tangent rule (Theorem 3.2). To find $\frac{\partial W}{\partial A}$, we apply the Leibnitz rule followed by observation that, by Theorem 3.2, T is a root of integrand.

In a similar way, we obtain

$$\frac{\partial W}{\partial \beta} = -\frac{l_\beta}{2} \text{ and } \frac{\partial W}{\partial \gamma} = -\frac{l_\gamma}{2}.$$

We note that $T \to \infty$ as $\alpha = \beta = \gamma \to \frac{\pi}{2}$. Then the condition $W(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$ follows from the convergence of integral.

□

Let $\Delta(\alpha, \theta)$ be defined by

$$\Delta(\alpha, \theta) = \Lambda(\alpha + \theta) - \Lambda(\alpha - \theta),$$

where $\Lambda(x)$ is the Lobachevsky function. We rewrite the previous theorem in the following way :

Theorem 3.4 (Kellerhals(1989), Mednykh(2003)) *The volume of a hyperbolic Lambert cube $Q(\alpha, \beta, \gamma)$ with essential angles $\alpha,\ \beta,\ \gamma\ (0 < \alpha, \beta, \gamma < \frac{\pi}{2})$ is given by the formula*

$$V(\alpha, \beta, \gamma) = \frac{1}{4}\left(\Delta(\alpha, \theta) + \Delta(\beta, \theta) + \Delta(\gamma, \theta) - 2\Delta(\frac{\pi}{2}, \theta) - \Delta(0, \theta)\right),$$

where $T = \tan\theta\ (0 < \theta < \frac{\pi}{2})$ is a root of the equation

$$T^4 - (A^2 + B^2 + C^2 + 1)T^2 - A^2B^2C^2 = 0,$$

$$A = \tan\alpha,\ B = \tan\beta,\ \textit{and}\ C = \tan\gamma.$$

In the spherical case, we have the following result obtained by Derevnin and Mednykh (2002).

Theorem 3.5 *The volume of a spherical Lambert cube $Q(\alpha, \beta, \gamma)$ with essential angles $\alpha, \beta, \gamma\ (\frac{\pi}{2} < \alpha, \beta, \gamma < \pi)$ is given by the formula*

$$V(\alpha, \beta, \gamma) = \frac{1}{4}\left(\delta(\alpha, \theta) + \delta(\beta, \theta) + \delta(\gamma, \theta) - 2\delta(\frac{\pi}{2}, \theta) - \delta(0, \theta)\right),$$

where

$$\delta(\alpha, \theta) = \int_\theta^{\frac{\pi}{2}} \log(1 - \cos 2\alpha \cos 2\tau)\frac{d\tau}{\cos 2\tau}$$

and $T = \tan\theta$ ($\frac{\pi}{2} < \theta < \pi$) *is a root of the equation*

$$T^4 - (A^2 + B^2 + C^2 + 1)T^2 - A^2B^2C^2 = 0,$$

$$A = \tan\alpha,\ B = \tan\beta,\ \text{and } C = \tan\gamma.$$

Remark 2 *The function* $\delta(x, \theta)$ *is considered as a spherical analogue of the function*

$$\Delta(x, \theta) = \Lambda(x + \theta) - \Lambda(x - \theta)$$

and satisfies the following properties :

1. $\delta(x, \theta)$ *is continuous for all* $(x, \theta) \in \mathbb{R}^3$ *and differentiable for* $x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$.

2. $\delta(x, 0) = \frac{\pi^2}{4} - \left|\frac{\pi^2}{2} - \pi x\right|,\ 0 \le x \le \pi$

3. *Let* $\widetilde{\delta}(x, \theta) = \delta(x, \theta) + (\frac{2\theta}{\pi} - 1)\delta(x, 0)$. *Then*

   (a) $\widetilde{\delta}$ *is even and* π*-periodic on* x

   (b) $\widetilde{\delta}$ *is odd and* π*-periodic on* θ

   (c) $|\widetilde{\delta}(x, \theta)| \le \frac{\pi^2}{4}$ *and* $\widetilde{\delta}(\frac{\pi}{2}, \frac{3\pi}{4}) = \frac{\pi^2}{4}$.

Remark 3 *The main result of R. Kellerhals(1989) for hyperbolic volume can be obtained from the Theorem 3.5 by replacing* $\delta(\alpha, \theta)$ *to* $\Delta(\alpha, \theta)$.

3.3 The volume of orthoscheme and Schläfli function

Let $T(\alpha, \beta, \gamma)$ be a double-rectangular tetrahedron (orthoscheme) with dihedral angles $\frac{\pi}{2} - \alpha,\ \beta$, and $\frac{\pi}{2} - \gamma$. See Fig.

Define a Schläfli function by the formula

$$S(\alpha, \beta, \gamma) = \left\{\sum_{n=1}^{\infty} \frac{(-X)^n}{n^2}(\cos 2n\alpha - \cos 2n\beta + \cos 2n\gamma - 1)\right\} - \alpha^2 + \beta^2 - \gamma^2,$$

where

$$X = \frac{\sin\alpha \sin\gamma - D}{\sin\alpha \sin\gamma + D},\ D = \sqrt{\cos^2\alpha \cos^2\gamma - \cos^2\beta},$$

$$0 \le \alpha \le \frac{\pi}{2},\ 0 \le \beta \le \pi,\ 0 \le \gamma \le \frac{\pi}{2}.$$

It was shown by Schläfli (1898) that in spherical case ($\cos^2\alpha \cos^2\gamma > \cos^2\beta$), the volume of $T(\alpha, \beta, \gamma)$ and $S(\alpha, \beta, \gamma)$ are related by

$$4Vol(T(\alpha, \beta, \gamma)) = S(\alpha, \beta, \gamma).$$

In Euclidean case ($\cos^2\alpha\cos^2\gamma = \cos^2\beta$), we have

$$S(\alpha,\beta,\gamma) = 0.$$

In hyperbolic case ($\cos^2\alpha\cos^2\gamma < \cos^2\beta$, and $\alpha < \beta,\ \gamma < \beta$), Coxeter (1935), using the results by Lobachevsky, has shown that

$$i\,S(\alpha,\beta,\gamma) = 4Vol(T(\alpha,\beta,\gamma)).$$

See Coxeter's paper (1935) for careful explanation of these result.

Now our aim is to prove the following theorem.

Theorem 3.6 (Derevnin, Mednykh, 2002) *Let $T = T(\alpha,\beta,\gamma)$ be a spherical orthoscheme with dihedral angles $\frac{\pi}{2}-\alpha,\ \beta,$ and $\frac{\pi}{2}-\gamma$ $(0 < \alpha,\beta,\gamma < \frac{\pi}{2})$. Then*

$$4Vol(T) = -\delta(\alpha,\theta) + \delta(\beta,\theta) - \delta(\gamma,\theta) + \delta(0,\theta),$$

where

$$\delta(\alpha,\theta) = \int_\theta^{\frac{\pi}{2}} \log(1-\cos 2\alpha\cos 2\tau)\frac{d\tau}{\cos 2\tau}$$

and

$$\tan\theta = \frac{\sin\alpha\sin\gamma}{\sqrt{\cos^2\alpha\cos^2\gamma - \cos^2\beta}}.$$

Proof. (1st step) We note that the following tangent rule takes place (Vinberg, p.125)

$$\frac{\tan\alpha}{\tan a} = \frac{\tan\beta}{\tan b} = \frac{\tan\gamma}{\tan c} = T,$$

where $T = \tan\theta = \frac{\sin\alpha\sin\gamma}{D}$, $D = \sqrt{\cos^2\alpha\cos^2\gamma - \cos^2\beta}$, and a, b, c are lengths of respective edges.

(2nd step) We need also the following cosine rule

$$\frac{\cos\beta}{\cos b} = \frac{\cos\alpha}{\cos a}\cdot\frac{\cos\gamma}{\cos c}.$$

The proof of this relation can be done in a pure geometric way. It also follows from the observation that T is a root of the equation

$$\frac{1+A^2}{T^2+A^2}\cdot\frac{T^2+B^2}{1+B^2}\cdot\frac{1+C^2}{T^2+C^2}\cdot T^2 = 1,$$

where

$$A = \tan\alpha,\ B = \tan\beta,\ C = \tan\gamma,$$

and a, b, c are lengths of respective edges. This rule, by virtue of tangent rule, is equivalent to

$$\frac{\cos^2\alpha}{\cos^2 a}\cdot\frac{\cos^2\beta}{\cos^2 b}\cdot\frac{\cos^2 c}{\cos^2\gamma}=1.$$

We note that the above equation has four roots $T_{1,2}=\pm 1$ and $T_{3,4}=\pm\frac{\sin\alpha\sin\gamma}{D}$.

(3rd step) The Schläfli formula.

Let $V = Vol(T(\alpha,\beta,\gamma))$ be the volume of $T(\alpha,\beta,\gamma)$. Then

$$\frac{\partial V}{\partial\alpha}=-\frac{a}{2},\ \frac{\partial V}{\partial\beta}=-\frac{b}{2},\ \frac{\partial V}{\partial\gamma}=-\frac{c}{2},$$

and $V\to 0$ as $T\to\infty$.

(4th step) We check that the function

$$W=\frac{1}{4}\int_T^\infty \log\frac{(1+A^2)(t^2+B^2)(1+C^2)t^2}{(t^2+A^2)(1+B^2)(t^2+C^2)}\frac{dt}{t^2-1}$$

satisfies the above system of differential equations. Then $W = V$. We have

$$\begin{aligned}4V &= \frac{1}{4}\int_T^\infty \log\frac{(1+A^2)(t^2+B^2)(1+C^2)t^2}{(t^2+A^2)(1+B^2)(t^2+C^2)}\frac{dt}{t^2-1}\\ &= -I(T,A)+I(T,B)-I(T,C)+I(T,0),\end{aligned}$$

where

$$I(T,A)=\int_T^\infty \log\frac{t^2+A^2}{1+A^2}\frac{dt}{t^2-1}.$$

Let $T=\tan\theta$. Under substitution $t=\tan\tau$, we obtain

$$\begin{aligned}I(T,A) &= \int_\theta^{\frac{\pi}{2}}\frac{\log(1-\cos 2\tau\cos 2\alpha)d\tau}{\cos 2\tau}-\int_\theta^{\frac{\pi}{2}}\frac{\log(1+\cos 2\tau)d\tau}{\cos 2\tau}\\ &= \delta(\alpha,\theta)-\delta(\frac{\pi}{2},\theta).\end{aligned}$$

Hence

$$4Vol(T(\alpha,\beta,\gamma))=-\delta(\alpha,\theta)+\delta(\beta,\theta)-\delta(\gamma,\theta)+\delta(0,\theta).$$

□

As a consequence of Theorem 3.6, we have the following form for the Schläfli function

$$S(\alpha,\beta,\gamma)=-\delta(\alpha,\theta)+\delta(\beta,\theta)-\delta(\gamma,\theta)+\delta(0,\theta).$$

In hyperbolic case, the following relation was obtained by Coxeter (1935)

$$i\,S(\alpha,\beta,\gamma)=-\Delta(\alpha,\theta)+\Delta(\beta,\theta)-\Delta(\gamma,\theta)+\Delta(0,\theta),$$

where

$$\tan\theta=\frac{\sin\alpha\sin\gamma}{\sqrt{\cos^2\beta-\cos^2\alpha\cos^2\gamma}}.$$

References

[1] Y. Wolf, Spaces of constant curvature, Publish or Perish, Berkeley, USA, 1972.

[2] James W. Cannon, William J. Floyd, Richard Kenyon, and Walter R. Parry, Hyperbolic Geometry, MSRI Publications, Volume 31, 1997.

[3] Y. Milnor, Collected papers vol.1, Publish or Perish, 1994, *The Schläfli differential equality*

[4] Vinberg, Mennicke, Helling, 1995. MR1968863(98i:20031)

[5] W. Thurston, Lectures on 3-manifold.

[6] C.D. Hodgson, Schläfli revisited : variation of volume in constant curvature space, preprint.